Math 1d	Instructor: Padraic Bartlett

Homework 7

Week 8 — *Caltech - Winter 2012*

Instructions: Choose **three** questions out of the following **five** to complete. If you attempt more than three questions, your three highest scores will be recorded as your grade. Also, write me/Daiqi an email if you get stuck!

Resources allowed: Wikipedia, Mathematica, your notes, the online class notes, textbooks, and your classmates. If you've completed a problem via Mathematica or writing a program, attach your code to receive credit.

Also: because of the short week and the esoteric/tricky nature of this HW set, this problem set is strictly optional! In other words, your HW grade is now going to be your best 5 scores out of the first 6 sets, with this set as an optional extra set which can replace any other score. Please, do attempt the problems – but if you're crushed by your other classes, or missed lecture on Wednesday and have no idea what's going on here, don't kill yourself to complete this set.

On the other hand, though, these problems are beautiful. So there's that.

More Interesting Exercises:

1. Using the fact that
$$x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 = x(x+1)(x^2+1)(x^4+1),$$
along with the fact that the three polynomials $(x+1), (x^2+1), (x^4+1)$ are all irreducible, decide whether or not there are a pair of nonstandard 8-dice that, when rolled and summed, are indistinguishable from a pair of standard 8-dice when rolled and summed.

2. (a) In class, we showed that if A, B are a pair of dice with associated polynomials $A(x), B(x)$, and $D(x)$ is the polynomial associated to rolling A and B and summing their result, we have
$$A(x) \cdot B(x) = D(x).$$
Mimicking this result, show that if A, B, C are three dice with associated polynomials $A(x), B(x), C(x)$, then
$$A(x) \cdot B(x) \cdot C(x) = D(x),$$
where $D(x)$ is the polynomial associated to rolling A, B, C and summing the result.

   (b) Using this result, decide whether or not there exists a triple A, B, C of non-standard 6-sided dice such that when they are rolled and summed, they are indistinguishable from rolling and summing three standard 6-sided dice.

3. Show that the polynomial
$$p(x) = 1 + x + x^2 + x^3 + x^4$$
is irreducible over the integer polynomials: in other words, it cannot be factored into two polynomials of strictly lower degree with integer coefficients.

   (Hint: what are the roots of this polynomial over $\mathbb{C}$? What happens to roots when you factor a polynomial into two other polynomials?)

4. Prove that there are no pairs of nonstandard 5-dice that when rolled and summed, are indistinguishable from a pair of standard 5-dice. (Hint: use problem (3). You may use problem 3 even if you did not prove it.)

5. As we noted in class, another approach to our dice question is simply to use a brute-force approach. Do this: in other words, create a computer program that

   - takes as input a number n, and
   - outputs all possible pairs of nonstandard n-dice, that when rolled and summed are indistinguishable from a pair of standard n-dice.

   (A rough outline: Simply run through the collection of all n dice with face values between 1 and 2n: for each pair, check whether rolling and summing that pair gives you the same probability distribution as rolling and summing a pair of standard n-dice. This will take *forever* – i.e like $C \cdot 11^{12}$ calculations to just go through all of the 6-sided dice – but it should work, and there's assuredly room for optimization if you did want to make this more viable.)