Math 1d	Instructor: Padraic Bartlett

Homework 5

Week 6 | *Caltech - Winter 2012*

Instructions: Choose **three** questions out of the following **six** to complete. If you attempt more than three questions, your three highest scores will be recorded as your grade. Also, write me/Daiqi an email if you get stuck!

Resources allowed: Wikipedia, Mathematica, your notes, the online class notes, textbooks, and your classmates. If you've completed a problem via Mathematica or writing a program, attach your code to receive credit.

Standard Exercises:

1. Let

$$f(x) = \ln(1 + x^2).$$

   (a) Find the Taylor series $T(f(x))$.

   (b) Look at the power series given by $T(f(x))$. What is its radius of convergence?

2. Like $\int e^{-x^2} dx$, the indefinite integral

$$\int \frac{\sin(x)}{x} dx$$

cannot be expressed as some combination of the elementary functions $\{\sin, \cos, e^x, \ln, \text{polynomials}\}$: i.e. there is no way to write the antiderivative of $\frac{\sin(x)}{x}$ by any combination of such elementary functions.

Despite this, use Taylor series to approximate the definite integral

$$\int_1^3 \frac{\sin(x)}{x} dx$$

to within $\pm.01$ of its exact value. (Hint: mimic the example in lecture where we approximated the integral $\int_{1/2}^{1/2} e^{-x^2} dx$.)

3. (a) Using integration by parts and an inductive argument, show that

$$\int_0^\infty e^{-at} \cdot t^n dt = \frac{n!}{a^{n+1}}.$$

   (b) Using the Taylor series for $\cos(t)$ and part (a), show that for $a > 1$, we have

$$\int_0^\infty e^{-at} \cos(t) dt = \frac{a}{a^2 + 1}.$$

   (Hint: By using (a) and $\cos(t)$, you'll be able to turn this integral into a power series evaluated at $1/a$. What power series is this?)

More Interesting Problems:

4. Recall, from the first problem set, the Fibonacci sequence:

$$f_0 = 0, f_1 = 1, f_{n+1} = f_n + f_{n-1}.$$

Look at the following power series you get by using the Fibonacci sequence, starting at f_1, as the coefficients of the x^n's: i.e.

$$\begin{aligned} P(x) =& \sum_{n=0}^{\infty} f_{n+1} \cdot x^n \\ =& 1 + 1x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21x^7 \ldots \end{aligned}$$

Show that this power series is the Taylor series of the function

$$f(x) = \frac{1}{1 - x - x^2}.$$

(Hint: Notice that $f(x)$ is defined by the equation $f(x) - xf(x) - x^2 f(x) = 1$.)

5. Sometimes, the Taylor series of a function can be useful even where it fails to converge! Consider the following example:

$$G(x) = \int_0^{\infty} e^{-x \cdot t} \cdot \frac{1}{1+t} dt.$$

If we were to attack this integral like problem 3, we would want to expand $\frac{1}{1+t}$ into its power series

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n \cdot t^n = 1 - t + t^2 - t^3 \ldots$$

However, we know that this series is only convergent for $|t| < 1$, and we're integrating on $(0, \infty)$.

(a) Despite this, do it anyways! I.e. show that the integral

$$\int_0^{\infty} e^{-x \cdot t} \cdot \left(1 - t + t^2 - t^3 \ldots\right) dt.$$

is the power series

$$P(x) = \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \frac{4!}{x^5} \ldots$$

(Hint: you are allowed to use 3(a) without proof.)

(b) Show that this series diverges for any nonzero $x \in \mathbb{R}$.

(c) However, surprisingly enough, we can still use this power series to roughly calculate $G(x)$! I.e. it is known that $G(10)$ is about .0915633.
Using Mathematica/matlab/a calculator, find the partial sums $P_n(10)$ of the first n terms in this power series evaluated at 10, for n between 1 and 14. Which value(s) of n are the closest to the actual value of this integral?

(This problem is a motivating example for the study of **asymptotic series**, which are a beautiful/beautifully complicated extension of the idea of Taylor series to "approximating things around infinity," rather than 0. Let me know if you want more information!)

6. Let $i = \sqrt{-1}$. Suppos that $P(x) = \sum_{n=0}^{\infty} a_n x^n$ is a power series with all $a_n \in \mathbb{R}$, such that

$$P(x) = P(ix), \forall x \in \mathbb{R}.$$

Show that $a_n = 0$ whenever n is not a multiple of 4.