Math 1d	Instructor: Padraic Bartlett

Homework 3

Week 4 — *Caltech - Winter 2012*

Instructions: Choose **three** questions out of the following **six** to complete. If you attempt more than three questions, your three highest scores will be recorded as your grade. Also, write me/Daiqi me an email if you get stuck!

Resources allowed: Wikipedia, Mathematica, your notes, the online class notes, textbooks, and your classmates. If you've completed a problem via Mathematica or writing a program, attach your code to receive credit.

Standard Exercises:

1. Determine whether the following series converge or diverge:

(a) $$\sum_{n=1}^{\infty}(-1)^n \cdot \sin\left(\frac{1}{n}\right).$$

(b) $$\sum_{n=1}^{\infty}\frac{\cos(n)\cdot n^3}{n!}.$$

(c) $$\int_{\pi}^{\infty}\frac{\sin(x)}{x}dx.$$

Hint: for (c), try writing this integral as the sum

$$\sum_{n=1}^{\infty}\int_{n\pi}^{(n+1)\pi}\frac{\sin(x)}{x}dx.$$

2. (a) Show that the pointwise limit of the sequence $\{f_n\}_{n=1}^{\infty}$, where

$$f_n(x) = \begin{cases} 0, & x \leq 0, \\ x^n - x^{2n}, & 0 < x < 1, \\ 0, & 1 \leq x, \end{cases}$$

is 0 (where we think of 0 as the function that always returns 0 on any input.)

(b) Does this sequence converge **uniformly** to 0? (Hint: take each f_n and find the value of x at which it takes on its maximum, using the derivative. What is f_n at that maximum value?)

3. (a) Create a sequence $\{f_n\}_{n=1}^{\infty}$ of discontinuous functions on $\mathbb{R}$ that converges **uniformly** to 0.

(b) Create a sequence $\{g_n\}_{n=1}^{\infty}$ of functions on $\mathbb{R}$ that converge **uniformly** to 0, but such that

$$\int_{-\infty}^{\infty} g_n(x)dx = 1, \forall n \in \mathbb{N}.$$

(Hint: in class, we showed that if a sequence $\{g_n\}_{n=1}^{\infty}$ converges uniformly to some function $g(x)$, then *on any finite interval* $[a, b]$, $\lim_{n\to\infty} \int_a^b g_n(x)dx = \int_a^b g(x)$. So on any given interval, the integrals of your functions is going to have to converge to 0: it's only by using all of $\mathbb{R}$ that you're going to be able to construct your sequence.) (Also, style points if you answer both (a) and (b) using the same sequence.)

More Interesting Problems:

4. In class, we rearranged the terms of the series$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ into the series

$$\begin{aligned} &\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\left(\frac{1}{7}-\frac{1}{14}\right)-\frac{1}{16}\cdots \\ =&\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\frac{1}{14}-\frac{1}{16}\cdots \\ =&\frac{1}{2}\cdot\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}. \end{aligned}$$

(a) Using similar techniques, rearrange and group/split up the terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ to get a series that converges to

$$2\cdot\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}.$$

(b) Now, create a rearrangement of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ into a series that **diverges**. (Hint: use the fact that the harmonic series diverges to show that if you add up a bunch of terms all with the same sign, you can get arbitrarily large sums. Use this to describe a rearrangement into a series that cannot converge.)

5. The following theorem (Dirichlet's test) is a generalization of the alternating series test we discussed on Monday:

Theorem 1 *(Dirichlet's test): Suppose that $\{b_n\}_{n=1}^{\infty}$ is a sequence such that the partial sums of the b_n's are bounded: i.e. there is some value M such that*

$$\lim_{n\to\infty}\left|\sum_{k=1}^{n} b_k\right| < M.$$

Suppose further that the sequence $\{a_n\}_{n=1}^{\infty}$ is such that

- $\lim_{n\to\infty} a_n = 0$,
- *the a_n's are all positive, and*
- *the a_n's are a monotonically decreasing sequence.*

Then the series

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

The alternating series test is a special case of this theorem, if we let $b_n = (-1)^{n+1}$. Using this theorem, prove that

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n}$$

converges.

(Hint: look at the sum $\sum_{k=1}^{n} \sin(k)\sin(1/2)$. Using the angle-addition formula $2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta) - \cos(\alpha+\beta)$, can you show this is bounded? How does this sum relate to the sum $\sum_{k=1}^{n} \sin(k)$? How does this observation help you apply Dirichlet's test?)

6. (Dynamical Systems!): Suppose that $f(x)$ is a continuous real-valued function on $\mathbb{R}$ with the following property:

$$\forall a, b \in \mathbb{R}, |f(a) - f(b)| < \frac{1}{2}|a - b|.$$

Let $\{f_n(x)\}_{n=1}^{\infty}$ be defined recursively by

$$f_1(x) = f(x),$$
$$f_{n+1}(x) = f(f_n(x)).$$

In other words, $f_n(x)$ is just the function created by composing f with itself n times.

(a) Using induction, show that for any value x, we have

$$|f_{n+1}(x) - f_n(x)| \leq \frac{1}{2^n} \cdot |f(x) - x|.$$

(b) Using a telescoping sum, write

$$f_{n+1}(x) = \sum_{k=1}^{n} f_{k+1}(x) - f_k(x).$$

Use this observation, part (a), and your knowledge of series to show that

$$\lim_{n\to\infty} f_{n+1}(x)$$

exists and is finite.

(c) Let x_0 denote the limit of the above process. Show that $f(x_0) = x_0$: i.e. that x_0 is a **fixed point** of f. (Hint: apply f to both sides of your limit, and use the fact that f is continuous.)

(d) Show that no matter what value of x we start the above process with, **you always get the same value** x_0**!** (Hint: suppose you have two fixed points, and apply f to both of them.)