Math 1d	Instructor: Padraic Bartlett

Homework 2

Week 3 | *Caltech - Winter 2012*

Instructions: Choose **three** questions out of the following **six** to complete. If you attempt more than three questions, your three highest scores will be recorded as your grade. Also, write me/Daiqi me an email if you get stuck!

Resources allowed: Wikipedia, Mathematica, your notes, the online class notes, textbooks, and your classmates. If you've completed a problem via Mathematica or writing a program, attach your code to receive credit.

Standard Exercises:

1. Determine whether the following series converge or diverge:

$$\text{(a)} \qquad \sum_{n=3}^{\infty} \frac{1}{n \ln(n)}.$$

$$\text{(b)} \qquad \sum_{n=3}^{\infty} \frac{1}{n \ln^2(n)}.$$

$$\text{(c)} \qquad \sum_{n=3}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}}.$$

   (Hint: Try using the integral test, and then doing integration by substitution. The same substitution will work for all three!)

2. Suppose that $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are a pair of sequences such that the series

$$\sum_{n=1}^{\infty} a_n^2, \qquad \sum_{n=1}^{\infty} b_n^2$$

   both converge.

   Prove that the series

$$\sum_{n=1}^{\infty} (a_n \cdot b_n)$$

   also converges. (Hint: In class, we used the comparison test to show that if $\sum a_n$ converged, then $\sum \frac{\sqrt{a_n}}{n}$ also converged. Can you use the methods in that proof here?)

3. In class, we proved that the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{n^n}$ converges.

(a) Using the ratio test and the identity $\lim_{n\to\infty}(1-\frac{1}{n})^n = \frac{1}{e}$, show that the series

$$\sum_{n=1}^{\infty} \frac{a^n \cdot n!}{n^{n+1}}$$

converges for $a < e$, and diverges for $a > e$.

(b) When $a = e$, the ratio test is inconclusive. Use the bound

$$\frac{n^n}{e^{n-1}} < n!$$

to show that the series

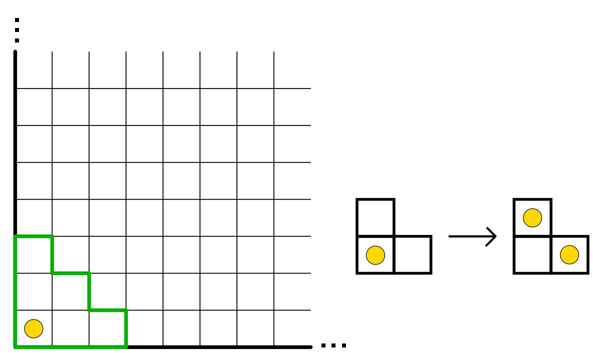
$$\sum_{n=1}^{\infty} \frac{e^n \cdot n!}{n^{n+1}}$$

diverges.

More Interesting Problems:

4. Suppose you have a $\mathbb{N} \times \mathbb{N}$ grid of 1×1 squares. Consider the following game you can play on this board:

   - Starting configuration: put one coin on the square in the bottom-right-hand corner of our board.
   - Moves: suppose that there is a coin on the board such that the squares immediately to its north and east are empty. Then a valid move is the following: remove this coin from the board, and then put one new coin on the north square and another new coin on the east square.



Is it possible to get all of the coins out of the green region above in a finite number of moves? Or will there always be coins in that region, no matter what you do?

Hint: one common way that mathematicians try to study games or models like this one is by finding an **invariant**, i.e. a function or quantity that we can associate to our system that doesn't change after moves are made. For example:

(a) Try to create a way to assign "weights" to every tile, so that the total weight of tiles with coins on them is an **invariant**. In other words, find a way to associate weights $w(i, j)$ to every tile (i, j), so that the sum of these weights over all of the coin-containing tiles doesn't change when we perform a move. (Try using negative powers of 2!)

(b) What is the total weight of everything outside of the green region? What is the total weight you start with? What can you deduce from this?

5. In class, we showed that the series $\sum \frac{1}{n}$ diverges. Show that the sum

$$\sum_{\substack{n \in \mathbb{N}: \\ n \text{ has no } 9 \\ \text{in its digits}}} \frac{1}{n}$$

converges to some value < 80. (Hint: For any k, how many k-digit numbers are there with no 9 in their digits? For any k-digit number n, what is the maximum size of $\frac{1}{n}$? How can you combine these two calculations to get an upper bound on our series?)

6. In class, we said that a sequence $\{a_n\}_{n=1}^{\infty}$ was **summable** if the sequence of its partial sums $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$ existed and was finite. This definition is the typical way to talk about summability: however, other mathematicians have introduced different concepts of what it means to "sum" an infinite sequence! One such example is the following:

Definition. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, and let

$$s_k = \sum_{n=1}^{k} a_n$$

denote the sum of the first k terms of our sequence. We say that the sequence $\{a_n\}_{n=1}^{\infty}$ is **Cesàro-summable** if the "average values" of its partial sums converge: i.e. if there is some A such that

$$\lim_{n\to\infty} \frac{1}{n} \cdot \left(\sum_{k=1}^{n} s_k \right) = A.$$

Remark/fact/thing you don't have to prove: If a sequence is summable in the normal sense, it's also Cesàro-summable. This is because if the partial sums converge to some value L, then the average value of the partial sums should also converge to L (because almost all of them are arbitrarily close to L.)

(a) Find a sequence that is not Cesàro-summable. Justify your claim.

(b) Find a sequence that **is** Cesàro-summable, but not summable in the normal sense. What does its Cesàro sum converge to?