MATH 1D, WEEK 6 – COMPLEX POWER SERIES AND THE RECIPROCALS OF SQUARES

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Abstract. These are the lecture notes from week 6 of Ma1d, the Caltech mathematics course on sequences and series.

1. HW 3 information

- Homework average: 85%.
- Issues: about 1/5 of the class completely blanked on the definition of uniform convergence, which put a dent in the overall average. Beyond that, however, that, everything looked fairly solid! See me if you're confused on any of the questions, or the concept of uniform convergence in general.

2. Complex Numbers: A Brief Review

Last quarter, the following question often came up: "Given some polynomial $P(x)$, what are its roots?" Depending on the polynomial, we had several techniques for finding these roots (Rolle's theorem, quadratic/cubic formulas, factorization;) however, we at times would encounter polynomials that possessed no roots at all, like

$$x^2 + 1$$

Yet, despite the observation that this polynomial's graph never crossed the x-axis, we *could* use the quadratic formula to find that this polynomial had the "formal" roots

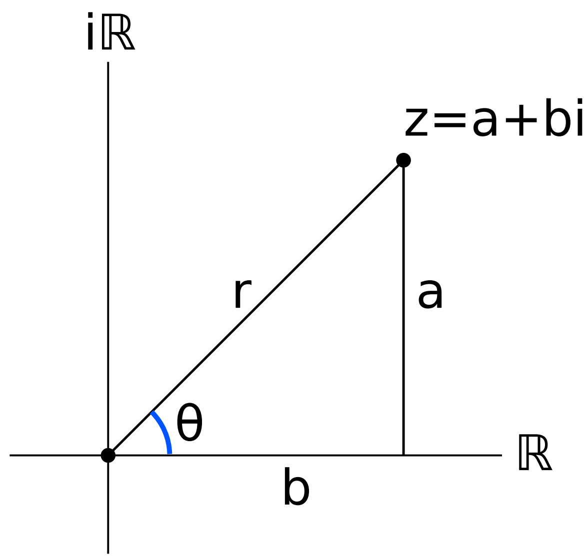
$$\frac{-0 \pm \sqrt{-4}}{2} = \pm\sqrt{-1}.$$

The number $\sqrt{-1}$, unfortunately, isn't a real number (prove this!) – so we had that this polynomial has no roots over $\mathbb{R}$. This was a rather frustrating block to run into; often, we like to factor polynomials entirely into their roots, and it would be quite nice if we could always do so, as opposed to having to worry about irreducible functions like $x^2 + 1$.

Motivated by this, we created the **complex numbers** by – essentially – just throwing $\sqrt{-1}$ into the real numbers. Formally, we defined the set of complex numbers, $\mathbb{C}$, as the set of all numbers $\{a + bi : a, b \in \mathbb{R}\}$, where $i = \sqrt{-1}$.

From this relatively simple definition came a massive host of properties, theorems, and definitions, which we review here briefly:

- We can graph complex numbers in the plane by mapping the value $a + bi$ to the point (a, b) in the plane, as done below:



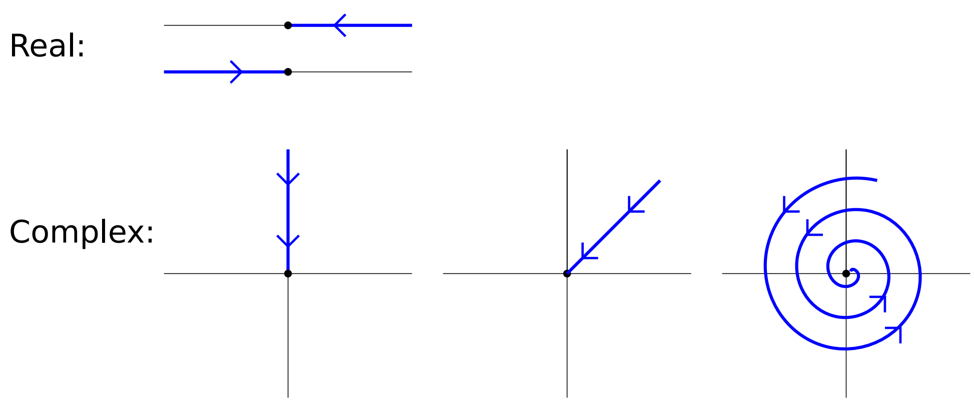
- This process of graphing complex numbers on the plane suggests that we might be able to associate other coördinate systems to the complex numbers; namely, polar coördinates! Specifically, we associated the pair (r, θ) to the point $z = a + bi$ as depicted above, and wrote

$$z = re^{i\theta}.$$

  The "why" of the above expression is not terribly clear right now: why should we say that the point z with polar coördinates (r, θ) *is* $re^{i\theta}$? Why would we apply the exponential function to i times the angle? On its face, there seems to be absolutely no good reason for doing this; yet, as it turns out, this expression is at the heart of one of the most elegant and beautiful equations in mathematics, and is – truly – the precise expression we would hope to be true. (Our goal for the first part of these lectures, incidentally, will be to prove this aforementioned equation: $e^{i\theta} = \cos(\theta) + i \sin(\theta)$, Euler's formula.)
- The derivative of a complex function: formally, we define the derivative of a complex function $f : \mathbb{C} \to \mathbb{C}$ at a point a by

$$f'(a) = \lim_{z \to 0} \frac{f(z + a) - f(a)}{z}.$$

  This, as you may have noticed, looks identical to the definition we had for the derivative of a real function! Yet, it is (unfortunately) a much more complicated beast, as the pictures below show:



In the real case, we're examining the limit $\lim_{h\to 0}$, where h is a real number; so, realistically, there are only two "paths" that we have to consider for studying this limit, $\lim_{h\to 0^-}$ and $\lim_{h\to 0^+}$. In the complex case, however, we have to deal with the limit $\lim_{z\to 0}$, where z is a complex number; in this case, we have an infinitude of possible paths that we have to consider, as the above diagrams show.

The study of complex differentiation and its deep connections to mathematics could take up its own course; for now, however, the sole point that you should take away from the above picture is that it is (in some well-defined way) **far harder** for a function to be complex-differentiable than it is to be real-differentiable.

- That said, a few of the basic theorems on differentiation still go through:
  - $f' = 0$, if f is a constant.
  - $f' = 1$, if $f(z) = z$.
  - $(f+g)' = f' + g'$.
  - $(fg)' = f'g + g'f$ – i.e. the product rule.
  - $(f \circ g)' = (f' \circ g) \cdot g'$ – i.e. the chain rule.

  For example, we still have that $z^n = nz^{n-1}$, by just applying the product rule and the property that $(z)' = 1$.
- So: we can also define things like sequences and series for complex numbers! In particular, the following definitions hold: for a sequence of complex numbers $\{a_n\}_{n=1}^{\infty}$,
  - We say that $\lim_{n\to\infty} a_n = l$ if $\lim_{n\to\infty} |a_n - l| = 0$.
  - We say that $\sum_{n=1}^{\infty} a_n = l$ if $\lim_{N\to\infty} \sum_{n=1}^{N} a_n = l$.
  - A complex power series around the point c is simply a complex valued function $f(z)$ of the form $\sum_{n=0}^{\infty} a_n(z-c)^n$.
- Our definitions for complex convergence, series, and power series look fairly similar to the ones we had for real series and power series; so, we might hope that some of our theorems carry through. Thankfully, many of them do, with special attention to the following theorem:

Theorem 2.1. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a complex power series that converges for some $z_0 \in \mathbb{C}$, then for any $a \le |z_0|$, we have that*
- *$\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on the circle of radius a in $\mathbb{C}$,*
- *the series $\sum_{n=1}^{\infty} a_n \cdot nx^{n-1}$ converges uniformly there as well, and*
- *$f'(x) = \sum_{n=1}^{\infty} a_n \cdot nx^{n-1}$ on this circle as well.*

The upshot of this theorem is that we are allowed to integrate and derive complex power series term-wise, just as we could with real power series! I.e. if $f(z) = \sum a_n z^n$ is a complex power series convergent over some region, then

- $f'(z) = \sum n a_n z^{n-1}$, and
- $\int f(z)dx = \sum \frac{a_n z^{n+1}}{n+1}$, up to a constant C.

- Finally: as alluded to in our "motivation" for the complex numbers, working in $\mathbb{C}$ solves many of our woes with respect to factoring out roots. In particular, we have the following theorem:

Theorem 2.2. *The Fundamental Theorem of Algebra: every complex polynomial $p(z)$ with degree n has n (possibly repeated) roots in the complex plane.*

As it turns out, there is a far stronger analogue to this theorem, which says (basically) that we can factor not just polynomials, but entire power series into their roots! We state this without proof below:

Theorem 2.3. *Weierstrass Factorization Theorem: every complex power series $f(x) = \sum a_n z^n$ can be written in the form*

$$e^{g(z)} x^k \cdot \prod_{\textit{all roots } r_i \textit{ of } f} \left(1 - \frac{z}{r_i}\right),$$

for some k and complex power series $g(x)$.

Basically, this says that we can separate any complex power series into its roots, times some $e^{g(z)}$-part that's never 0. In particular, we have that

$$\sin(z) = z \cdot \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{\pi n}\right),$$

by factoring it into its roots.

The proof of this theorem – or indeed just that $\sin(z)$ can be written in the form above! – are far beyond the scope of this course. But it should hopefully be somewhat believable to you all that this is plausible; after all, if we can factor out the roots for polynomials, then we ought to be able to do so for "infinte polynomials" like power series.

(A quick aside: for those of you who haven't seen it before, the **infinite product** of some sequence a_n, $\prod_{n=1}^{\infty} a_n$, is just defined by the limit

$$\lim_{N\to\infty} \prod_{n=1}^{N} a_n.$$

This should hopefully look identical to the definition we used for an infinte series; it's basically the same idea, except with multiplication in place of addition.)

3. Examples of Functions on $\mathbb{C}$

All of that said: the only functions we've defined for complex numbers are polynomials, thus far. It would be nice to perhaps actually *have* some function to work with; say, the trigonometric functions, or e^z, or log!

Well; the issue here is that extending, say, $\sin(x)$ to the whole complex plane is kind of a weird thing to do. Where should we send $\sin(i)$ to, for example? On the face of things, it's not remotely clear.

However, we *do* know how to extend polynomials to the complex plane: we just send x^n to z^n, and get what arguably is the only natural interpretation of what x^n could be in the complex numbers. So: motivated by this idea, we actually can choose to **define** $\sin(z), \cos(z)$, and e^z by their power series!

In particular: because

$$\begin{aligned}\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots,\\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots, \text{and}\\ e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\end{aligned}$$

for all real x, we choose to define

$$\begin{aligned}\sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \dots,\\ \cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \dots, \text{and}\\ e^z &= 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \dots,\end{aligned}$$

for all $z \in \mathbb{C}$.

In particular, we can see by examining e^{iz} that

$$\begin{aligned}e^{iz} &= 1 + iz + \frac{(iz)^2}{2} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \dots\\ &= 1 + iz - \frac{z^2}{2} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} \dots\\ &= \left(1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} \dots\right)\\ &= \cos(z) + i\sin(z);\end{aligned}$$

in other words, that $e^{iz} = \cos(z) + i\sin(z)$. This equation is known as Euler's formula, and is considered one of the most beautiful theorems in the natural sciences – plugging in $z = \pi$, in particular, tells us that $e^{i\pi} - 1 = 0$, an equation linking pretty much every single interesting mathematical constant in one succinct equality.

A quick application of Euler's equation is the following: first, notice that any point with polar coördinates (r, θ) can be written in the plane as $(r\cos(\theta), r\sin(\theta))$. This tells us that any point with polar coördinates (r, θ) in the complex plane, specifically, can be written as $r(\cos(\theta) + i\sin(\theta))$; i.e. as $re^{i\theta}$. Thus, as it turns out, our seemingly arbitrary decision to write points in the form $re^{i\theta}$ is in fact the *only* choice we have left to us, if we're going to define things like $\sin, \cos$, or e in the complex plane at all!

We can also use the above ideas to define a kind of idea of $\log(z)$ via its power series $\log(1-z) = \sum_{n=1} \frac{z^n}{n}$ as well; however, another method of extension is also fairly natural, and easy to set up. In the real setting, we defined log and e to be inverses of each other; so it would seem natural to try to do the same thing here. So, given a number $re^{i\theta}$, we write $\log(re^{i\theta})$ as $\log(r) + i\theta$. This definition has the advantage of making sense on all of $\mathbb{C}$ except for the origin (as opposed to our power series, which only converges for $|z| < 1$); however, it bears noting that this defintion has some interesting quirks, like mapping the entire complex plane to the strip $\{a + bi : b \in (-\pi, \pi]\}$ and not being continuous. For a further discussion, check out the Wikipedia entry, or see me for more sources.

4. Proving $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$

Theorem 4.1. $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

Proof. So: recall our earlier-mentioned *deus ex machina* result that $\sin(z)$ could be "factored into its roots" – i.e that

$$\sin(z) = z \cdot \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{\pi n}\right).$$

We can rewrite this expression as the product

$$\sin(z) = z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{\pi n}\right) \cdot \prod_{n=1}^{\infty} \left(1 + \frac{z}{\pi n}\right),$$

and bring terms together to further simplify this into the equation

$$\begin{aligned} \sin(z) &= z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{\pi n}\right) \cdot \left(1 + \frac{z}{\pi n}\right) \\ &= z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right); \end{aligned}$$

Thus, from the above, we know that we can write

$$\frac{\sin(z)}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right).$$

Ok, so enough simplification. Why do we do this? Well: we also have the power series expansion $z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \ldots$, which tells us that

$$\begin{aligned} \frac{\sin(z)}{z} &= \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \ldots}{z} \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \frac{z^8}{9!} - \ldots \end{aligned}$$

So these two quantities are the same! In particular, we know that they must share the same power series (as shown by you all, as a class, on the fourth HW set;) consequently, these two objects must share the same z^2-coefficient. For the power series expression, finding this coefficient is easy – it's just $-\frac{1}{3!}$.

For the product, it's not much harder. Consider, in fact the infinite product

$$\left(1-\frac{z^2}{\pi^2 1^2}\right)\cdot\left(1-\frac{z^2}{\pi^2 2^2}\right)\cdot\left(1-\frac{z^2}{\pi^2 3^2}\right)\cdot\left(1-\frac{z^2}{\pi^2 4^2}\right)\cdot\ldots.$$

How can we get a term involving z^2 out of such a product? Well; think way back, to the days of FOIL. In particular, how do we figure out what a complicated finite product of polynomials like

$$(a_0 + a_1 z + \ldots a_q z^q)\cdot(b_0 + \ldots b_r z^r)\cdot(c_0 + \ldots + c_s z^t)$$

is? We just pick a term in the first polynomial – say, some $a_i z^i$ – and multiply it by some term in the second polynomial – say, $b_k z^k$ – and finally multiply it by some term in the third polynomial – say $c_l z^l$. If we do this exactly once for every single possible way of choosing terms out of these three polynomials, and add them up, this gives us the product of the polynomials! Essentially, this is just FOIL writ large.

So, in the infinte case it's just the same – in order to figure out what the terms of

$$\left(1-\frac{z}{\pi^2 1^2}\right)\cdot\left(1-\frac{z}{\pi^2 2^2}\right)\cdot\left(1-\frac{z}{\pi^2 3^2}\right)\cdot\left(1-\frac{z}{\pi^2 4^2}\right)\cdot\ldots$$

are, we just need to look at the various terms we get by choosing one value from each $(1-\frac{z^2}{\pi^2 n^2})$ and multiplying them all together. In particular, if we're looking at the z^2 coefficient, the only terms that will have a z^2 as their coefficient are those that choose precisely one $\frac{z^2}{\pi^2 n^2}$ out of our giant product, and choose 1's the rest of the time! So, in short, we have that the z^2 terms are simply all of the fractions $-\frac{1}{\pi^2 n^2}$; so the z^2-coefficient is just

$$\sum_{n=1}^{\infty} -\frac{1}{\pi^2 n^2}.$$

So: setting this equal to $-\frac{1}{3!}$ then tells us at last that

$$\begin{aligned} -\frac{1}{3!} &= \sum_{n=1}^{\infty} -\frac{1}{\pi^2 n^2} \\ \Rightarrow \qquad \frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2}, \end{aligned}$$

thus completing our proof. $\square$