# MATH 1D, WEEK 5 – UNIFORM CONVERGENCE AND POWER SERIES

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Abstract. These are the lecture notes from week 5 of Ma1d, the Caltech mathematics course on sequences and series.

## 1. Midterm information

- Midterm average: 79%.
- Good things: People, on the whole, did rather well! In particular, I was impressed with the style people displayed in their proofs; a number of students really stepped up their mathematical game for this test.
- However, there were a few issues: in particular, people didn't seem very comfortable with some of the basic definitions. There were more than a few tests that confused series and sequences, and a plurality that forgot how to deal with conditionally convergent series altogether; given that the test was open-book and open-notes, this was a little worrisome.
- That said, as a whole, the class exceeded my expectations. We've covered a huge amount of material through the first four weeks of this course; a retention rate of about 80%, considering how fast we've been moving, is wonderful. Good job!

## 2. Uniform Convergence

So: last class, we discussed what it might mean for a sequence of functions to "converge" – towards this goal, we came up with two possible definitions of convergence, which are reprinted below for your convenience:

**Definition 2.1.** For a sequence of functions $\{f_n\}$, from some set $A$ to $\mathbb{R}$, we say that

$$\lim_{n\to\infty} f_n = f \text{ pointwise}$$

if for every $x$ in $A$, we have that

$$\lim_{n\to\infty} f_n(x) = f(x).$$

**Definition 2.2.** For a sequence of functions $\{f_n\}$, from some set $A$ to $\mathbb{R}$, we say that

$$\lim_{n\to\infty} f_n = f \text{ uniformly}$$

if for every $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that whenever $n > N$, we have

$$|f_n(x) = f(x)| < \epsilon$$

for every $x$ in $A$.

Both definitions have their own advantages and disadvantages; as we showed in the last lecture, pointwise convergence is an easier condition to check and study, while uniform convergence has the advantage of conserving a number of qualities that we "like" in functions. Specifically, we showed that uniform convegence preserved the properties of continuity and the integral, in certain well-defined ways.

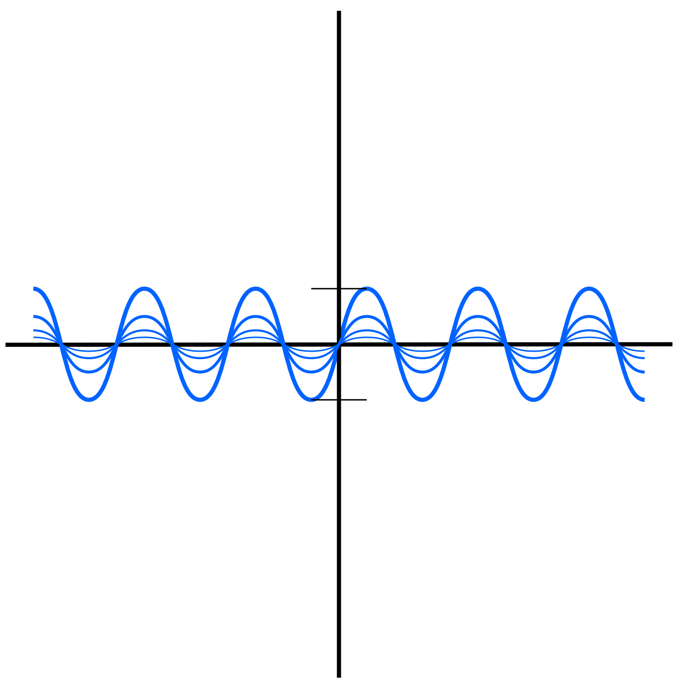
Given the results of our last lecture, then, a natural question to ask here is whether uniform convergence preserves other properties of functions: namely, whether uniform convergence preserves the concept of differentiability. To answer this question, we first consider the following pair of examples:

**Example 2.3.** Suppose that

$$f_n(x) = \frac{1}{n} \cdot \sin(n^2 x).$$

Does the sequence $\{f_n\}$ uniformly converge? If so, where does it converge to? Also, what happens to the derivatives of these functions at 0?

*Proof.* We first graph the $f_n$'s, to better illustrate what's going on here:



So: from the picture above, it seems very likely that these functions are uniformly converging to zero.

To prove this, simply take any $\epsilon$ greater than zero, and let $N > \frac{1}{\epsilon}$. Then, for any $n > N$, we have that

$$|f_n(x) - 0| = \left|\frac{1}{n}\sin(n^2 x)\right| = \frac{1}{n} \cdot \left|\sin(n^2 x)\right| \leq \frac{1}{n} < \frac{1}{N} < \epsilon;$$

so the sequence $f_n$ uniformly converges to 0, as we claimed.

The derivatives at zero, however, are far less well-behaved: in fact, we have that

$$\begin{aligned}\lim_{n\to\infty}(f_n)'(0) &= \lim_{n\to\infty} n\cdot\cos(n^2x)\Big|_0\\ &= \lim_{n\to\infty} n\cdot\cos(0)\\ &= \lim_{n\to\infty} n\\ &= \infty,\end{aligned}$$

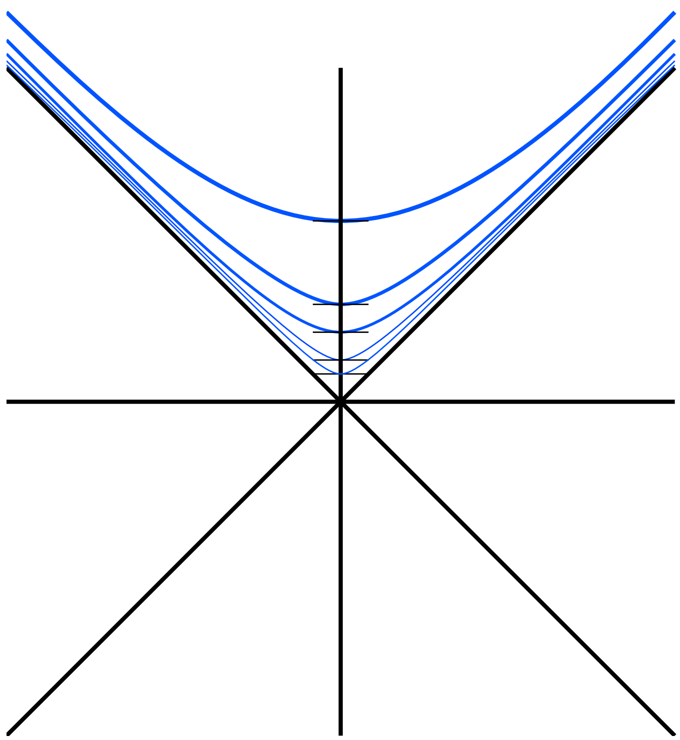
whereas the derivative of 0 at 0 is merely 0. So the derivatives here were not convserved – i.e. the limit of the derivative of the functions was not equal to the derivative of the limit! □

**Example 2.4.** Suppose that

$$f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$$

Does the sequence $\{f_n\}$ uniformly converge? If so, where does it converge to? Also, what happens to the derivatives of these functions at 0?

*Proof.* Again, we graph the $f_n$'s to better show what's going on here:



So: the graphs above are just those of the upper branch of the hyperbola $y^2-x^2 = \frac{1}{n^2}$; as such, we can visually "see" that these graphs will converge uniformly to the graph of their asymptotes, $|x|$.

To prove this, simply notice that for any $x, n$,

$$\begin{aligned} x^2 \leq x^2 + \frac{1}{n^2} &\leq x^2 + \frac{1}{n^2} + 2\frac{\sqrt{x^2}}{n^2} \\ \Rightarrow \sqrt{x^2} \leq \sqrt{x^2 + \frac{1}{n^2}} &\leq \sqrt{x^2} + \sqrt{\frac{1}{n^2}} \\ \Rightarrow |x| \leq \sqrt{x^2 + \frac{1}{n^2}} &\leq |x| + \frac{1}{n}; \end{aligned}$$

so, if we let $N > \frac{1}{\epsilon}$ we will again have that

$$|f_n(x) - |x|| \leq \left| |x| + \frac{1}{n} - |x| \right| = \frac{1}{n} < \epsilon$$

and thus that these functions uniformly converge to $|x|$.

However, again, we have that the derivatives are not conserved at zero! I.e. we explicitly have that

$$(f_n)'(0) = \left. \frac{2x}{2\sqrt{x^2 + \frac{1}{n^2}}} \right|_0 = 0,$$

but $|x|$ doesn't have a derivative at zero! So, again, the limit of the derivatives is not the derivative of the limits. □

These initial results might seem somewhat discouraging for our hopes of preserving any notion of the derivative. However, it bears noting that both of these sequences "failed" us in the same way – specifically, in both cases, we had the limit of the derivatives of the $f_n$'s converging to something discontinuous and otherwise ill-behaved. In fact, it turns out that this is the only way in which differentiability can not be conserved – i.e. if we force the derivatives of the $f_n$'s to converge to something continuous, then the derivatives of the $f_n$'s in fact converge to the derivative of their limit! The following theorem simply states this:

**Theorem 2.5.** *If the limit* $\lim_{n\to\infty} f_n = f$ *uniformly, and the limit* $\lim_{n\to\infty} f'_n$ *converges uniformly to some continuous function, then* $f$ *is differentiable and* $\lim_{n\to\infty} f'_n(x) = f'(x)$.

*Proof.* So; because the function $\lim_{n\to\infty} f'_n$ converges uniformly, we have that

$$\begin{aligned} \int_a^x \lim_{n\to\infty} f'_n(t)dt &= \lim_{n\to\infty} \int_a^x f'_n(t)dt \\ &= \lim_{n\to\infty} f_n(x) - f_n(a) \\ &= f(x) - f(a). \end{aligned}$$

Then, because $\lim_{n\to\infty} f'_n(x)$ is continuous, we can use the fundamental theorem to conclude that $\lim_{n\to\infty} f'_n(x) = f'(x)$. □

## 3. Power Series

Just like with sequences of numbers, it turns out that a large part of why we study sequences of functions is to better understand series of functions. Explicitly, we make the following definition:

**Definition 3.1.** For a sequence of functions $f_n$ from some set $A$ to $\mathbb{R}$, we say that the sum $\sum_{n=0}^{\infty} f_n$ **converges uniformly** to some function $f : A \to \mathbb{R}$ if and only if the sequence $\left\{\sum_{n=0}^{N} f_n\right\}_{N=1}^{\infty}$ converges uniformly to $f$.

So: we have a quick triple of theorems, the proofs of which are immediate from our proofs of the same results for sequences (and are thus omitted:)

**Theorem 3.2.** *Suppose that the sum $\sum_{n=0}^{\infty} f_n$ converges uniformly to some function $f : [a, b] \to \mathbb{R}$. Then,*

- *If all of the $f_n$'s are continuous on $[a, b]$, then so is $f$.*
- *If all of the $f_n$'s are integrable on $[a, b]$,then $\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$.*
- *If the sum $\sum_{n=1}^{\infty} f_n'$ converges to a continuous function on $[a, b]$, then $f' = \sum_{n=1}^{\infty} f_n'$.*

So: the above theorems are amazingly powerful, but getting to **use** them, at the moment, is something of a hassle; while we know a number of things about a series once we've established that it's uniformly convergent, we really don't have any tools beyond the definitions to show that a sequence **is** uniformly convergent. The theorem below, thankfully, will change that for us:

**Theorem 3.3.** *(Weierstrass M-test:) Suppose that $\{f_n\}$ is a sequence of functions from some set $A$ to $\mathbb{R}$, and $\{M_n\}$ is a sequence of numbers such that $|f_n(x)| \le M_n, \forall x \in A$, with the following properties:*

- *$|f_n(x)| \le M_n$, for every $x$ in $A$, and*
- *$\sum_{n=1}^{\infty} M_n$ converges absolutely.*

*Then the sum $\sum_{n=1}^{\infty} f_n$ converges uniformly.*

*Proof.* First, note that for every $x \in A$,

$$\sum_{n=1}^{\infty} |f_n(x)| \le \sum_{n=1}^{\infty} M_n < \infty$$

and thus (by the comparison test) we have that $\sum_{n=1}^{\infty} |f_n(x)|$ converges absolutely, for every $x \in A$. Denote the function that $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to by $f(x)$.

Thus, we have that for every $N \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{\infty} |f(x) - f_1(x) - f_2(x) - \ldots - f_N(x)| &\le \left|\sum_{n=N+1}^{\infty} f_n(x)\right| \\ &\le \sum_{n=N+1}^{\infty} |f_n(x)| \\ &\le \sum_{n=N+1}^{\infty} M_n.\end{aligned}$$

Because $\sum_{n=1}^{\infty} M_n$ converges absolutely, we know that this quantity $\sum_{n=N+1}^{\infty} M_n$ goes to 0 as $N$ goes to infinity; so, for any $\epsilon > 0$ we can pick a $N$ such that for every $n > N$,

$$\sum_{n=1}^{\infty} |f(x) - f_1(x) - f_2(x) - \ldots - f_n(x)| < \epsilon.$$

As this is precisely the condition for uniform convergence, we've thus shown that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly. $\square$

We will use this theorem mostly to study when certain series of functions, called **power series**, converge. To illustrate how this goes, we first define what a power series is, and work several example problems discussing their convergence:

**Definition 3.4.** A **power series around the point** $a$ is a series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot (x-a)^n,$$

where the $a_n$ are a collection of real numbers. Typically, we will be concerned with power series around the point 0, which take the form

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot x^n.$$

**Example 3.5.** Suppose that

$$f(x) = \sum_{n=0}^{\infty} x^n.$$

Where does this series converge? How does it converge?

*Proof.* First, note that if $|x| \geq 1$, we have that $\lim_{n\to\infty} x^n \neq 0$, and thus this series cannot possibly converge. So it suffices to consider the case when $x$ lies in the interval $(-1, 1)$; in this situation, we have that

$$|f(x)| = \left|\sum_{n=0}^{\infty} x^n\right| = \left|\frac{1}{1-x}\right| < \infty$$

by using our summation identity for the geometric series.

This, furthermore, proves that on any interval $[-a, a] \subset (-1, 1)$, the sum $\sum_{n=0}^{\infty} x^n$ converges uniformly to $\frac{1}{1-x}$, as on any such interval we have

$$|x^n| \leq a^n.$$

Because $\sum_{n=1}^{\infty} a^n$ converges, we can then apply the Weierstrass M-test to show that the sum $\sum_{n=0}^{\infty} x^n$ must converge uniformly on $[-a, a]$. $\square$

**Example 3.6.** Suppose that

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$$

Where does this series converge? How does it converge?

*Proof.* First, note that if $|x| > 1$, we have that $\lim_{n\to\infty} \frac{(-1)^n}{n} x^n \neq 0$, because $x^n >> n$ as $n$ grows large. Consequently, in those cases, $f$ cannot possibly converge; so it suffices to consider the case when $x$ lies in the interval $[-1, 1]$.

If $x = 1$, we have that $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$, which we have shown several times in the past to be convergent (via Leibniz's theorem.)

If $x = -1$, we have that $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \cdot (-1)^n = \sum_{n=1}^{\infty} \frac{1}{n}$, which we know to be the harmonic series and is thus divergent.

Finally, if $x \in (-1, 1)$, we have that

$$|f(x)| = \left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n \right| \leq \sum_{n=1}^{\infty} \frac{1}{n} |x|^n \leq \sum_{n=1}^{\infty} |x|^n = |x| \cdot \sum_{n=0}^{\infty} |x|^n = \frac{|x|}{1 - |x|} < \infty$$

by again using our summation identity for the geometric series; so this converges for all $x \in (-1, 1)$.

So this series converges for all $xin(-1, 1]$. As well, by the same argument as before, on any interval $[-a, a] \subset (-1, 1]$, we have

$$\left| \frac{(-1)^n}{n} x^n \right| \leq a^n$$

and also that the sum $\sum_{n=1}^{\infty} a^n$ converges; so, again by the Weierstrass M-test, the sum $\sum_{n=0}^{\infty} x^n$ converges uniformly on $[-a, a]$. □

**Example 3.7.** Suppose that

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n!} x^n$$

Where does this series converge? How does it converge?

*Proof.* So: simply note that by the ratio test, we have that

$$\lim_{n\to\infty} \frac{x^{n+1}/(n+1)!}{x^n/n!} = \lim_{n\to\infty} \frac{x}{n+1} = 0,$$

and thus that this series converges for every $x$ in $\mathbb{R}$. As well, just as in the two proceeding examples, we have that this converges uniformly on every single subset $[-a, a] \subset \mathbb{R}$, by combining the observations that

$$\left| \frac{x^n}{n!} \right| \leq \frac{a^n}{n!}$$

and that the sum $\sum_{n=1}^{\infty} a^n$ converges (via the ratio test), along with the Weierstrass M-test. □

**Example 3.8.** Suppose that

$$f(x) = \sum_{n=1}^{\infty} n! \cdot x^n$$

Where does this series converge? How does it converge?

*Proof.* So: again note that by the ratio test, we have that

$$\lim_{n\to\infty} \frac{x^{n+1} \cdot (n+1)!}{x^n \cdot n!} = \lim_{n\to\infty} x(n+1) = \infty,$$

for any $x \neq 0$. So this series only converges for $x = 0$. □

By glancing at the examples above, a few patterns might leap out at you; (1) that all of the power series above were convergent in symmetric intervals around the origin (up to the endpoints), and (2) that if a power series converges on a set, then it uniformly converges on any closed interval lying strictly inside of that set. At the least, that's what we saw in the examples above; we had power series converging on the sets $(-1,1), (-1,1], \mathbb{R}$, and 0, and in every case these series uniformly converged on every closed interval $[-a,a]$ lying within those sets.

It turns out this is actually true in general, for any power series! The following theorem demonstrates this result, along with another result which will hopefully illustrate just why we do all of this work:

**Theorem 3.9.** *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, and suppose that $x_0$ is a number in $\mathbb{R}$ such that $f(x_0)$ converges; i.e. such that $\sum_{n=0}^{\infty} a_n x_0^n$ converges.*

*Pick any $a$ such that $0 < a < x_0$. Then, we have that*

- *on $[-a,a]$, $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly,*
- *on $[-a,a]$, the series $\sum_{n=1}^{\infty} a_n \cdot nx^{n-1}$ converges uniformly as well, and*
- *$f'(x) = \sum_{n=1}^{\infty} a_n \cdot nx^{n-1}$ on all of $[-a,a]$.*

Basically, our theorem claims the following: if a power series converges at one point $x_0$, it converges uniformly on every interval $[-a,a]$ with $a < x_0$, and – furthermore – we can calculate its derivative by simply differentiating the power series itself termwise. This second statement is perhaps the most remarkable and useful result of this theorem, as it essentially makes the basic operations of calculus on any function given by a power series **completely trivial**! – because it lets us simply perform derivations on the function's power series, which is just a giant polynomial (and, consequently, is something we can handle with ease.)

*Proof.* So: because $\sum_{n=1}^{\infty} a_n x_0^n$ converges, we have that the terms $a_n x_0^n$ go to zero; thus, they must be bounded! Pick a bound $M$ such that $|a_n x_0^n < M$, for all $n$.

Then, if $x \in [-a,a]$, we have that

$$\begin{aligned} |a_n x^n| &= |a_n| \cdot |x^n| \\ &\leq |a_n| \cdot |a^n| \\ &= |a_n| \cdot |x_0^n| \cdot \left| \frac{a^n}{x_0^n} \right| \\ &\leq M \cdot \left| \frac{a}{x_0} \right|^n \end{aligned}$$

Because $a < x_0$, we know that $\left| \frac{a}{x_0} \right| < 1$; thus, the series

$$\sum_{n=0}^{\infty} M \cdot \left| \frac{a}{x_0} \right|^n = M \cdot \sum_{n=0}^{\infty} \left| \frac{a}{x_0} \right|^n$$

is geometric, and thus converges! So, by the Weierstrass M-test, taking $M_n = M \cdot \left| \frac{a}{x_0} \right|^n$, we know that $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[-a,a]$.

To see that the series $\sum_{n=1}^{\infty} a_n \cdot nx^{n-1}$ converges, we perform mostly the same trick: again, for any $x \in [-a, a]$, we have that

$$\begin{aligned}
|na_n x^{n-1}| &= n|a_n| \cdot |x^{n-1}| \\
&\leq n|a_n| \cdot |a^{n-1}| \\
&\leq \frac{n|a_n| \cdot |a^n|}{|a|} \\
&= \frac{n|a_n| \cdot |x_0^n| \cdot \left|\frac{a^n}{x_0^n}\right|}{|a|} \\
&\leq \frac{n \cdot M \cdot \left|\frac{a}{x_0}\right|^n}{|a|}.
\end{aligned}$$

Thus, because $\left|\frac{a}{x_0}\right| < 1$, the sum

$$\sum_{n=1}^{\infty} \frac{n \cdot M \cdot \left|\frac{a}{x_0}\right|^n}{|a|} = \frac{M}{|a|} \sum_{n=1}^{\infty} n \cdot \left|\frac{a}{x_0}\right|^n$$

converges via the ratio test (as we have shown several times). Applying the Weierstrass M-test, we can then conclude that this series converges uniformly as well! Finally, because uniform convergence preserves the derivative, we can see at last that this means that $f'(x)$ is actually given by this power series, $\sum_{n=1}^{\infty} a_n \cdot nx^{n-1}$. $\square$

To illustrate a little bit of the power inherent in the above theorem, here are a pair of examples:

**Example 3.10.** Using only the observations, from our earlier work with Taylor series, that

$$\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \ldots, \\
e^x &= 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots,
\end{aligned}$$

calculate the derivatives and antiderivatives of these functions.

*Proof.* So: by the above theorem, we know that taking derivatives or antiderivatives of these functions can be done just on their power series. Explicitly, then, we have that

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\dots\right) \\
&= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \frac{9x^8}{9!}\dots \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}\dots \\
\Rightarrow \frac{d^2}{dx^2}\sin(x) &= \frac{d}{dx}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}\dots\right) \\
&= -\frac{2x^1}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \frac{8x^7}{8!}\dots \\
&= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!}\dots \\
&= -\sin(x).
\end{aligned}$$

Thus, we've shown – using no properties of sin other than its power series – that its second derivative is just $-\sin(x)$, and consequently that its fourth derivative is just $\sin(x)$ again. As well, we've found the power series for its derivative; so if we knew $\sin(x)$'s power series, but not $\cos(x)$'s, this method just told us what it was!

As for $e^x$: because

$$\begin{aligned}
\frac{d}{dx}e^x &= \frac{d}{dx}\left(1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) \\
&= \frac{1}{1} + \frac{2x^1}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \dots \\
&= 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
&= e^x,
\end{aligned}$$

we can see that taking derivatives or antiderivatives of $e^x$ won't change it! So, again, from just its power series, we've been able to derive properties about $e^x$ which pretty much completely define it as a function. □