MATH 1D, HW #3 – QUESTIONS

INSTRUCTOR: PADRAIC BARTLETT

Instructions: Choose **three** questions out of the **four** below to complete! Also, justify everything you claim. Some of these are difficult questions! Write me if you have any questions.

Question 0.1. If $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series, prove that

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|.$$

Question 0.2. Find the pointwise limit f of the sequence of functions

$$f_n := \begin{cases} e^{-n}, & x < -1, \\ e^{-nx^2}, & -1 \leq x \leq 1, \\ e^{-n}, & x > 1. \end{cases}$$

Do the f_n converge uniformly to this f?

Question 0.3. Find the pointwise limit f of the sequence of functions

$$f_n := \begin{cases} 0, & x < 0, \\ x^n - x^{2n}, & 0 \leq x \leq 1, \\ 0, & x > 1. \end{cases}$$

Do the f_n converge uniformly to this f? (Hint: where does the function $x^n - x^{2n}$ adopt its maximum? What value does it take there?)

Question 0.4. Show that any rational number can be written as a finite sum of distinct numbers of the form $1/n$.

Date: Due Date: Thursday, Feb. 4, at 4 p.m.

For an idea on how to do question 4, consider the following algorithm for breaking up $\frac{29}{24}$ into fractions of the form $1/n$: because

$$\begin{aligned}
\frac{29}{24} - \frac{1}{2} &= \frac{17}{24}\\
\frac{17}{24} - \frac{1}{3} &= \frac{9}{24}\\
\frac{9}{24} - \frac{1}{4} &= \frac{3}{24}\\
\frac{3}{24} &< \frac{1}{5}\\
\frac{3}{24} &< \frac{1}{6}\\
\frac{3}{24} &< \frac{1}{7}\\
\frac{3}{24} - \frac{1}{8} &= 0,
\end{aligned}$$

we have that $\frac{29}{24}$ can be written as $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{8}$.

Explicitly, notice that the numerators in our fractions above are always decreasing! So, one proof of this theorem could be completed as follows:

- Take any rational number $\frac{p}{q}$ such that $\frac{1}{n+1} < \frac{p}{q} < \frac{1}{n}$, for some n. Show that the numerator of $\frac{p}{q} - \frac{1}{n+1}$ is strictly smaller than the numerator of $\frac{p}{q}$.
- Conclude that, because the numerators are decreasing, that a generalized form of the process above will always end in finitely many steps. This then tells us that $\frac{p}{q}$ can be written as the sum of finitely many distinct fractions $\frac{1}{k}$.
- Use the fact that the sum $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to show that any rational number – not just those between $\frac{1}{n+1}$ and $\frac{1}{n}$ for some n – can be written as the sum of finitely many such fractions $\frac{1}{k}$.