MATH 1D, WEEK 4 – THE ORDER OF SUMMATION; CONVERGENCE CONCEPTS FOR FUNCTIONS

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Abstract. These are the lecture notes from week 4 of Ma1d, the Caltech mathematics course on sequences and series.

1. Homework 1 data

- HW average: 88%.
- Comments: as always: write more! Also, be careful that you're actually answering the question at hand; these problems were tricky enough that some of you forgot what you were proving part of the way through.

2. The Order of Summation

The last few lectures of this class have largely concerned themselves with discussing the conditions under which infinte sums converge. In all of these discussions, we've uniformly assumed that the infinte sum $\sum_{n=1}^{\infty} a_n$ denotes the limit

$$\lim_{n\to\infty} a_1 + a_2 + \ldots a_n.$$

In the above sum, we assume that when we take this limit, we are always adding up the a_n's "in order" – i.e. we don't look at the limit

$$\lim_{n\to\infty} a_1 + a_2 + a_4 + a_3 + a_6 + a_8 + a_5 + a_{10} + a_{12} + \ldots a_{2n-1} + a_{4n-2} + a_{4n},$$

where we're adding up one odd term for every two even terms.

So: a natural question to ask here is "why don't we?" After all, in the case of finite sums, the order of addition doesn't matter at all: e.g. $1+2+3 = 2+1+3 = 3+2+1$, regardless of how you add it up. So: does this extend to the case of infinte sums?

To partially answer this question, consider the sequence

$$\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \ldots$$

The terms of this series alternate in sign and go to zero: therefore, by Leibniz's theorem, we know that this series converges. So: suppose that we could rearrange the terms of this sum without altering what it adds up to. Then, we would have (in particular, by using the rearrangement suggested above) that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \ldots \\
&= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} \ldots \\
&= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \frac{1}{16} \ldots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} \ldots \\
&= \frac{1}{2} \cdot \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \ldots\right) \\
&= \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n}.
\end{aligned}$$

Since the only solution to the equation $x = \frac{1}{2}x$ is for x to be zero, we 've just shown that the sum above must be zero if we're allowed to rearrange the terms in its sum. But, we also know that

$$\begin{aligned}
\Rightarrow \sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \ldots \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) \ldots \\
&= \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \frac{1}{7 \cdot 8} + \ldots \\
&> \frac{1}{2} > 0;
\end{aligned}$$

so this sum is most definitely nonzero! So, at least in the case of the series above, rearranging terms seems to radically change the sum of the original series.

For a moment, we might hope that the above result was just some artifact of the construction we chose, or the specific series we were working with. This, unfortunately, is very far from the truth, as the following theorem demonstrates:

Theorem 2.1. *Pick any series $\sum_{n=1}^{\infty} a_n$ that converges conditionally, and choose any $r \in \mathbb{R}$. Then there is a rearrangement of the sequence$\{a_n\}_{n=1}^{\infty}$ into a sequence $\{b_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} b_n = r$.*

(In other words: if $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent sequence, then by switching its terms around we can make it sum up to any number we want.)

Proof. Before we begin, we should define what we mean by a "rearrangement" of a sequence:

Definition 2.2. A **rearrangement** of a sequence $\{a_n\}$ is another sequence $\{b_n\}$ that takes the same values as the a_n's as many times as the a_n's do. Explicitly, a rearrangement of the sequence $\{a_n\}$can be thought of as a sequence $\{b_n\}$ and a bijective function $f : \mathbb{N} \to \mathbb{N}$ such that

$$a_{f(n)} = b_n, \forall n.$$

So: to begin our proof, take any conditionally convergent series $\sum_{n=1}^{\infty} a_n$, and suppose without any loss of generality that the constant r we want to rearrange $\sum_{n=1}^{\infty} a_n$ to sum to is positive. (The proof will look the exact same if $r \leq 0$.)

Given our series $\sum_{n=1}^{\infty} a_n$, define the following two sequences $\{p_n\}_{n=1}^{\infty}, \{q_n\}_{n=1}^{\infty}$ as follows:

$$\begin{aligned} \{p_n\} &:= \text{ the positive terms of the } a_n\text{'s} \\ \{q_n\} &:= \text{ the negative terms of the } a_n\text{'s.} \end{aligned}$$

For example, if $\{a_n\}_{n=1}^{\infty}$ was the sequence $\left\{\frac{(-1)^{n+1}}{n}\right\}$, then

$$\begin{aligned} \{p_n\} &:= \left\{\frac{1}{2n-1}\right\}_{n=1}^{\infty}, \text{ and} \\ \{q_n\} &:= \left\{-\frac{1}{2n}\right\}_{n=1}^{\infty}. \end{aligned}$$

So: from week 3's notes from Thursday, we know that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if and only if the sums $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ are both convergent. So, because we're assuming that $\sum_{n=1}^{\infty} |a_n|$ is divergent, we then know that at least one of the sums $\sum_{n=1}^{\infty} p_n$ or $\sum_{n=1}^{\infty} q_n$ must diverge.

We claim that in fact both of these sums must diverge. To see why, simply proceed by contradiction; i.e. assume that $\sum_{n=1}^{\infty} p_n$ converges and $\sum_{n=1}^{\infty} q_n$ diverges (the case where $\sum_{n=1}^{\infty} p_n$ diverges but $\sum_{n=1}^{\infty} q_n$ converges is handled in a similar way). However, this means that we have

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \lim_{N\to\infty}\left(\sum_{n=1}^{N} a_n\right) \\ &= \lim_{N\to\infty}\left(\sum_{\substack{\text{all } a_n>0,\\ n\leq N}} a_n + \sum_{\substack{\text{all } a_n<0,\\ n\leq N}} a_n\right) \\ &\leq \lim_{N\to\infty}\left(\sum_{n=1}^{\infty} p_n + \sum_{\substack{\text{all } a_n<0,\\ n\leq N}} a_n\right) \\ &= \sum_{n=1}^{\infty} p_n + \lim_{N\to\infty}\left(\sum_{\substack{\text{all } a_n<0,\\ n\leq N}} a_n\right) \\ &= \sum_{n=1}^{\infty} p_n + \lim_{N\to\infty}\left(\sum_{n=1}^{N} q_n\right) \\ &= -\infty. \end{aligned}$$

But $\sum a_n$ is conditionally convergent; so this is a contradiction! Hence, both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ must both diverge.

The remainder of the proof here is a relatively simple idea that requires an unfortunate amount of notation to describe. For the busy reader, the basic idea is precisely the same as the proof that we used for question 4 on homework 2 – i.e. because the sums of the positive terms and the sums of the negative terms both diverge, we can just "add up" positive terms until we get bigger than r, then "add up" negative terms until we become smaller than r, and repeat this process until we've added up all of the terms in $\{a_n\}$. Because the terms a_n go to 0, we know that these partial sums must converge to r; so we're done!

To make the above formal: because $\sum_{n=1}^{\infty} p_n$ diverges, we know that there is some M_1 such that $\sum_{n=1}^{M_1} p_n > r$. Pick the smallest such M_1 such that this is true: i.e pick M_1 such that

$$\sum_{n=1}^{M_1-1} p_n \leq r, \sum_{n=1}^{M_1} p_n > r.$$

Note further that, by the construction above, that M_1 in some sense marks the "best approximation" to r that we can get by adding up the values of the a_n's; i.e. that

$$\left| \sum_{n=1}^{M_1} p_n - r \right| < p_{M_1}.$$

Denote the quantity $\left| \sum_{n=1}^{M_1} p_n - r \right|$ by the symbol S_1.

Then, because $\sum_{n=1}^{\infty} q_n$ also diverges, we know that there is some value N_1 such that $\sum_{n=1}^{M_1} p_n + \sum_{n=1}^{N_1} q_n < r$; again, pick the smallest value such that this holds. This, again, puts us in the situation such that

$$\sum_{n=1}^{M_1} p_n + \sum_{n=1}^{N_1-1} q_n \geq r, \sum_{n=1}^{M_1} p_n + \sum_{n=1}^{N_1} q_n < r.$$

Again, because of our construction above, we can think of N_1 as another "best approximation" to r, in that

$$\left| \sum_{n=1}^{M_1} p_n + \sum_{n=1}^{N_1} q_n - r \right| < |q_{N_1}|.$$

Denote this quantity $\left| \sum_{n=1}^{M_1} p_n \sum_{n=1}^{N_1} q_n - r \right|$ by the symbol T_1.

Repeat the process above to generate a series of values M_i, N_i, S_i, T_i for all $i \in \mathbb{N}$ such that the value M_i, N_i correspond to increasing "better approximations" to r and S_i, T_i continue to measure the distance between these partial sums and r.

So: take the rearrangement

$$p_1, p_2, \ldots, p_{M_1}, q_1, q_2, \ldots q_{N_1}, p_{M_1+1}, p_{M_1+2}, \ldots, p_{M_2}, q_{N_1+1}, q_{N_1+2}, \ldots, q_{N_2}, \ldots$$

given to us by our construction above. By definition, we know that the partial sums of the values in this sequence are bounded by the consecutive values of the S_i's and T_i's; but these values are both bounded (as we showed above for S_1, T_1) by p_{N_i} and q_{n_i}. As these values p_{N_i}, q_{n_i} are just increasingly further-along entries in the sequence a_n, and the a_n converge to zero (because the sum $\sum_{n=1}^{\infty} a_n$ converges,) we know that they must converge to zero – thus, the partial sums converge to r! So we've managed to rearrange the terms of the a_n's in such a way to make their corresponding series sum up to r, for any r. □

A natural question to ask after seeing the above proof is whether this is something that afflicts all sums; i.e whether we can do this "rearranging" trick on any series to get it to go wherever we would like. Intuitively, this seems impossible; i.e. if we add up the fractions $\frac{1}{2^n}$, it seems reasonable to expect that we'd always make it to 1, regardless of the order that we add them up in. This turns out to be true! i.e. we have the following theorem:

Theorem 2.3. *If the sum $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and $\{b_n\}$ is a rearrangement of the a_n's, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$.*

Proof. So: because the sum $\sum_{n=1}^{\infty} |a_n|$ converges, we know that for any $\epsilon >$ there is a N such that

$$\sum_{n=N+1}^{\infty} |a_n| < \epsilon.$$

Basically, this tells us that almost all of the "mass" of the sum $\sum_{n=1}^{\infty} |a_n|$ lies in its first N entries. Thus, if we look at partial sums of the a_n's and b_n's that contain these first N entries, we can see that they're basically the same, up to some ϵs. Letting ϵ go to zero then tells us that these sums converge to the same limit, and thus completes our proof!

More explicitly: as a result of the above, we have that

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} a_n\right| &\leq \left|\sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n\right| \\
&\leq \left|\sum_{n=1}^{N} a_n\right| + \sum_{n=N+1}^{\infty} |a_n| \\
&\leq \left|\sum_{n=1}^{N} a_n\right| + \epsilon, \text{ and} \\
\left|\sum_{n=1}^{\infty} a_n\right| &\geq \left|\sum_{n=1}^{N} a_n\right|.
\end{aligned}$$

Pick M to be a sufficiently large number such that all of the entries $a_1 \ldots a_N$ show up among the entries $b_1 \ldots b_M$. Then, we have that

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} b_n\right| &\leq \left|\sum_{n=1}^{N} a_n + \sum_{\substack{b_n\text{'s left over,}\\ n \leq N}} b_n + \sum_{n=N+1}^{\infty} a_n\right| \\
&\leq \left|\sum_{n=1}^{N} a_n\right| + \sum_{\substack{b_n\text{'s left over,}\\ n \leq N}} |b_n| + \sum_{n=M+1}^{\infty} |b_n| \\
&\leq \left|\sum_{n=1}^{N} a_n\right| + \sum_{n=N+1}^{\infty} |a_n| + \sum_{n=N+1}^{\infty} |a_n| \\
&\leq \left|\sum_{n=1}^{N} a_n\right| + 2\epsilon, \text{ and} \\
\left|\sum_{n=1}^{\infty} b_n\right| &\geq \left|\sum_{n=1}^{N} a_n\right|.
\end{aligned}$$

So the distance between these two sums is at most 2ϵ; letting ϵ go to zero then tells us that these sums converge to the same value. $\square$

3. Convergence and Functions

Thus far, all of our discussions about convergence have dealt with real numbers: over the last three and a half weeks, we've developed a number of theorems and tests designed to let us know when various sequences and series of real numbers converge, and to tell us what they converge to. Yet, the innate concept of convergence is just one of "distance" – essentially, the claim that a sequence converges to a value is just a way of saying that its terms become very "close" to that value.

So: if the key idea of convergence is just this idea of "distance," perhaps we can extend this concept of convergence to other objects. After all, we can formulate definitions of distance for all sorts of objects, not just real numbers; so why couldn't we come up with notions of convergence as well? This, in fact, is the aim of today's lecture: to come up with a few ideas of what convergence might mean for **functions** on the real numbers.

First, note that by a **sequence** of functions we will merely mean a collection $\{f_n\}_{n=1}^{\infty}$ of functions, indexed by the natural numbers. In this situation, suppose that all of the functions f_n are maps from some set A to the real numbers, and suppose further that we're given a function $f : A \to \mathbb{R}$. What could we possibly hope to mean by the equation

$$\lim_{n\to\infty} f_n = f \text{ ?}$$

One possible idea – certainly one of the most intuitive ideas – would be to simply say that $\lim_{n\to\infty} f_n = f$ holds if and only if the sequences $\{f_n(x)\}$ converge to $f(x)$, for every $x \in A$. In other words, we'd be making the definition that a sequence of functions converges to some value iff it converges at every point; a sort of "point-wise" notion of convergence, if you will. This is in fact what this notion

– **pointwise convergence** – is called; to emphasize its importance, we define it more formally here:

Definition 3.1. We say that a sequence $\{f_n\}$ of functions $A \to \mathbb{R}$ **converges pointwise** to some function $f : A \to \mathbb{R}$ if and only if $\lim_{n\to\infty} f_n(x) = f(x)$, for every $x \in A$.

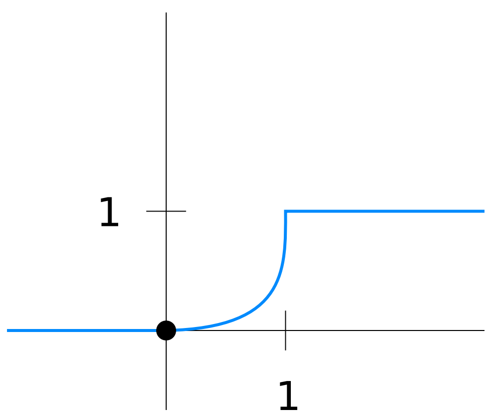
So: if a sequence of real numbers all had a certain property – like all being positive, or greater than three, or integers – then if they converged to some value, that value often had to share that property. A natural question, then, is whether this holds true for sequences of functions; in other words, if we have a sequence of differentiable/continuous functions, must their pointwise limit be differentiable/continuous? If we have a sequence of functions all with integral 1 over some region, must their pointwise limit? We seek to answer these questions via a series of examples, calculated below:

Example 3.2. Let

$$f_n(x) := \begin{cases} 0, & x < 0 \\ x^n, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

What is the pointwise limit of the f_n's?

Proof. Before we begin, we offer a graph of one of the f_n's, to hopefully motivate our calculations below:



To calculate what the $f_n(x)$'s converge to, we break the x's apart into three cases:

(1) $x < 0$. In this case, we have that
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0.$$
(2) $0 \le x < 1$. In this case, we have that
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0,$$
as well.

(3) $1 \ge x$. In this case, we have that
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 1 = 1.$$

Combining the results above tell us that the functions f_n converge pointwise to the function

$$f(x) := \begin{cases} 0, & x < 1 \\ 1, & x \geq 1 \end{cases}$$

□

Interestingly enough, the function above isn't continuous, despite the fact that all of the f_n's were! So pointwise convergence doesn't seem to be "enough," in a sense, to force continuous things to stay continuous.

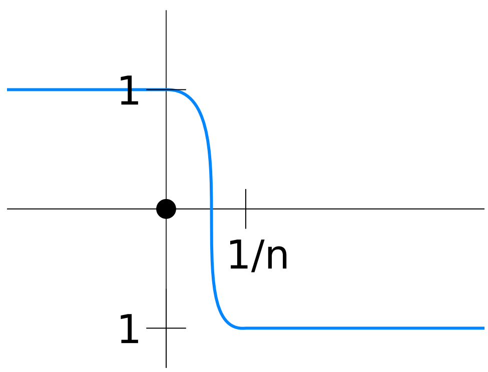
Perhaps this was an artifact of the fairly stilted and disjointed construction of the f_n's above in our first example; i.e. maybe if we picked a sequence of f_n's that were "smoother" – say, differentiable – they would stay continuous when we passed to the pointwise limit. We explore this hypothesis in the following example:

Example 3.3. Let

$$f_n(x) := \begin{cases} 1, & x \leq 0 \\ \cos(n\pi x), & 0 < x < 1/n \\ -1, & x \geq 1/n \end{cases}$$

What is the pointwise limit of the f_n's?

Proof. We again offer a motivational graph of the f_n's:



Before beginning, we note that these functions indeed are all differentiable, as their derivatives on each part of their piecewise definition are

$$f_n'(x) := \begin{cases} 0, & x \leq 0 \\ -n\pi \cdot \sin(n\pi x), & 0 < x < 1/n \\ 0, & x \geq 1/n \end{cases},$$

and these all agree at the "cross-over" points 0 and $1/n$.

So: to calculate what these $f_n(x)$'s converge to, we just break the x's apart into two cases:

(1) $x \leq 0$. In this case, we have that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 1 = 1.$$

(2) $x > 0$. In this case, we know (from the first quarter) that we can always find a value of N such that $\frac{1}{N} < x$; thus, for every $n > N$, we have that $f_n(x) = -1$, because $x > \frac{1}{n}$. Thus, we have that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} -1 = -1.$$

Combining the results above tell us that the functions f_n converge pointwise to the function

$$f(x) := \begin{cases} 1, & x \leq 0 \\ -1, & x > 0 \end{cases}$$

□

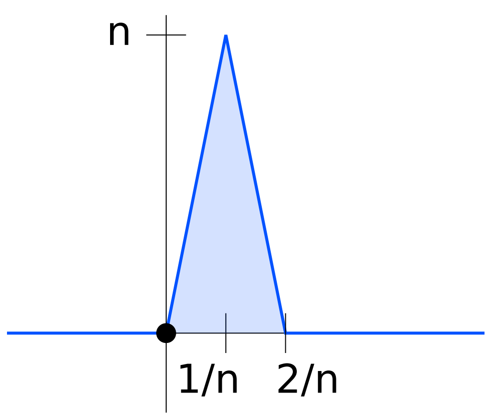
Apparently, not even choosing differentiable curves can help us preserve continuity (much less differentiability itself!) So, the last hypothesis we have left to explore is whether the integral is preserved under pointwise limits – i.e. if I have a sequence of functions f_n all with integral $\int_0^\infty f_n(x)dx = 1$, say, then must their pointwise limit f also have $\int_0^\infty f(x)dx = 1$? Based on our bleak results thus far, you may be able to guess the conclusion we draw from our third example, below:

Example 3.4. Let

$$f_n(x) := \begin{cases} 1, & x \leq 0 \\ n^2x, & 0 < x \leq 1/n \\ -n^2x + 2n, & 1/n < x \leq 2/n \\ 0, & x \geq 2/n \end{cases}$$

What is the pointwise limit of the f_n's?

Proof. We once again open with a graph of the f_n's:



By construction, the integral of any of these f_n's is just the area of a triangle with base $2/n$ and height n – i.e. 1, for every f_n.

So: to calculate what these $f_n(x)$'s converge to, we again break the x's apart into two cases:

(1) $x \leq 0$. In this case, we have that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0.$$

(2) $x > 0$. In this case, we again know that we can find a value of N such that $\frac{2}{N} < x$; thus, for every $n > N$, we have that $f_n(x) = 0$, because $x > \frac{2}{n}$. Thus, we have that

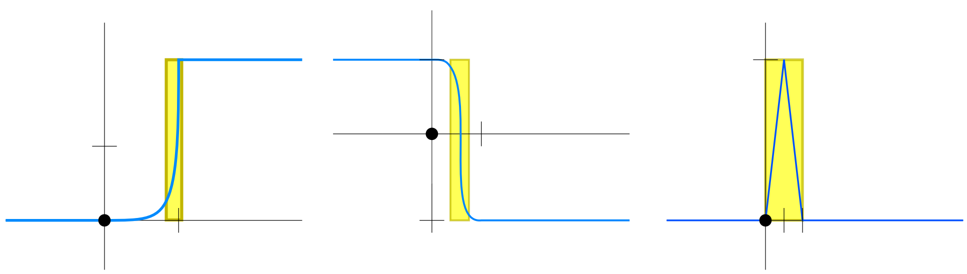
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0.$$

Combining the results above tell us that the functions f_n converge pointwise to the function

$$f(x) := 0.$$

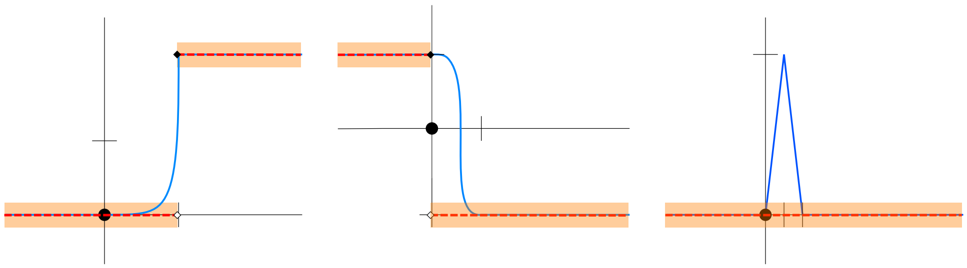
□

The moral of the above three examples seems to be that our notion of pointwise convergence, as intuitive and easy-to-use as it is, fails miserably at conserving most of the basic concepts we have for describing functions. Continuous functions fail to stay continuous, integrals aren't stable, differentiability has no hope; it's all a big mess. Yet, by looking at the graphs of the three "counterexamples" above, we can perhaps come up with a remedy for this dilemma:



In each of the three graphs above, there's a region (highlighted in yellow) where the graph seems to be almost moving "too fast" – i.e. while all of the f_n's remain continuous for every n, as the n's get large our functions begin to have massive displacement over a very small area (as in the yellow regions.) So, while the f_n's converged pointwise to their pointwise limits f, at any point along their convergence there always remained a small region – corresponding to the yellow areas – where these functions were very **far** apart.

So: what if we used this as a new notion for convergence? I.e. what if we said that a sequence of functions f_n converged to a function f if and only if the f_n's became **uniformly** close to the function f? In other words: what if we said that $\lim_{n\to\infty} f_n = f$ if and only if the f_n's are eventually ϵ-close to f everywhere, for any epsilon and large enough n? Well, we definitely wouldn't have to worry about our three earlier examples, as the picture below shows:



Here, we can see that the f_n's never lie within a small neighborhood (say, the one shaded in orange) of their pointwise limits f: so, while they do converge to f pointwise, they would fail under our proposed definition above! So, there's maybe some merit to this idea: so let's formally define this notion of a "uniform" convergence, and see where it takes us:

Definition 3.5. We say that a sequence $\{f_n\}$ of functions $A \to \mathbb{R}$ **converges uniformly** to some function $f : A \to \mathbb{R}$ if and only if for every $\epsilon > 0$, there is a N such that for every $n > N$,

$$|f(x) - f_n(x)| < \epsilon, \forall x \in A.$$

In other words, a sequence $\{f_n\}$ converges uniformly to some function f if and only if the f_n's are all $\epsilon-$close to f **everywhere**, for sufficiently large n.

The payoff for this definition lies in the following two theorems, which simply state that uniform convergence preserves continuity and integrals. We state and prove them below:

Theorem 3.6. *If* $\lim_{n\to\infty} f_n = f$ *uniformly, and all of the functions* f_n, f *are integrable on some interval* $[a, b]$*, then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Proof. So: because $\lim_{n\to\infty} f_n = f$ uniformly, we know that (by definition) for any $\epsilon > 0$ there is a N such that for all $n > N$, $|f(x) - f_n(x)| < \epsilon$ for all $x \in [a, b]$.

But this means that for all $n > N$,

$$\begin{aligned}\left|\int_a^b f(x) - \int_a^b f_n(x)dx\right| &= \left|\int_a^b (f(x) - f_n(x))\, dx\right| \\ &\leq \int_a^b |f(x) - f_n(x)|\, dx \text{ (we proved this property of integrals last quarter)} \\ &\leq \int_a^b \epsilon dx \\ &= (b-a)\epsilon.\end{aligned}$$

Because we can pick ϵ to be arbitrarily small, we have that $\int_a^b f(x) = \int_a^b f_n(x)dx$, as claimed. □

Theorem 3.7. *If* $\lim_{n\to\infty} f_n = f$ *uniformly, and all of the functions* f_n *are continuous on some interval* (a, b)*, then so is* $f(x)$.

Proof. So: because $\lim_{n\to\infty} f_n = f$ uniformly, we know that (by definition) for any $\epsilon > 0$ there is a n such that , $|f(x) - f_n(x)| < \epsilon/3$ for all $x \in [a, b]$.

In particular, for any h such that both $x, x + h$ lie in (a, b), we have that

$$|f(x) - f_n(x)| < \epsilon/3$$
$$|f(x + h) - f_n(x + h)| < \epsilon/3.$$

But we also know by the continuity of f_n that for every ϵ, there is a δ such that if $|h| < \delta$,

$$|f_n(x) - f_n(x+h)| < \epsilon/3.$$

Adding these three inequalities together, we have that for any $\epsilon > 0$, there is a δ such that for any x in (a, b) and $|h| < \delta$,

$$
\begin{aligned}
& |f(x) - f_n(x)| + |f_n(x) - f_n(x+h)| + |f_n(x+h) - f(x+h)| < \epsilon/3 + \epsilon/3 + \epsilon/3 \\
\Rightarrow \quad & |f(x) - f_n(x) + f_n(x) - f_n(x+h) + f_n(x+h) - f(x+h)| < \epsilon \\
\Rightarrow \quad & |f(x) - f(x+h)| < \epsilon.
\end{aligned}
$$

But this is the literal definition of continuity for $f(x)$! Thus, we have shown that uniform convergence preserves continuity, as claimed. □