MATH 1D, WEEK 3 – THE RATIO TEST, INTEGRAL TEST, AND ABSOLUTE CONVERGENCE

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ABSTRACT. These are the lecture notes from week 3 of Ma1d, the Caltech mathematics course on sequences and series.

1. Homework 1 data

- HW average: 91%.
- Comments: none in particular – people seemed to be pretty capable with this material.

2. The Ratio Test

So: thus far, we've developed a few tools for determining the convergence or divergence of series. Specifically, the main tools we developed last week were the two comparison tests, which gave us a number of tools for determining whether a series converged by comparing it to other, known series. However, sometimes this isn't enough; simply comparing things to $\sum \frac{1}{n}$ and $\sum r^n$ usually can only get us so far. Hence, the creation of the following test:

Theorem 2.1. *If $\{a_n\}$ is a sequence of positive numbers such that*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r,$$

for some real number r, then

- *$\sum_{n=1}^{\infty} a_n$ converges if $r < 1$, while*
- *$\sum_{n=1}^{\infty} a_n$ diverges if $r > 1$.*

If $r = 1$, this test is inconclusive and tells us nothing.

Proof. The basic idea motivating this theorem is the following: if the ratios $\frac{a_{n+1}}{a_n}$ eventually approach some value r, then this series eventually "looks like" the geometric series $\sum r^n$; consequently, it should converge whenever $r < 1$ and diverge whenever $r > 1$.

To make the above concrete: suppose first that $r > 1$. Then, because $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$,, we know that there is some $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\frac{a_{n+1}}{a_n} > 1$$
$$\Rightarrow a_{n+1} > a_n.$$

But this means that the a_n's are eventually an increasing sequence starting at some value $a_N > 0$; thus, we have that $\lim_{n\to\infty} a_n \neq 0$ and thus that the sum $\sum_{n=1}^{\infty} a_n$ cannot converge.

Conversely: suppose that $r < 1$. Then, again by definition, we can find a $N \in \mathbb{N}$ and a s such that $1 > s > r$ with the property that for all $n \geq N$,

$$\begin{aligned} \frac{a_{n+1}}{a_n} &< s \\ \Rightarrow a_{n+1} &> s \cdot a_n. \end{aligned}$$

Specifically, this means that

$$\begin{aligned} a_N &< s \cdot a_{N+1} \\ \Rightarrow a_N &< s \cdot a_{N+1} < s \cdot s \cdot a_{N+2} \\ \Rightarrow a_N &< s \cdot a_{N+1} < s^2 \cdot a_{N+2} < s^2 \cdot s \cdot a_{N+3} \\ \Rightarrow \ldots \\ \Rightarrow a_N &< s^k \cdot a_{N+k}, \end{aligned}$$

(where the conclusion above comes from a fairly trivial induction.)

But this means that

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n \\ &\leq \sum_{n=1}^{N-1} a_n + \sum_{n=0}^{\infty} s^n \cdot a_N \\ &= \sum_{n=1}^{N-1} a_n + a_N \cdot \sum_{n=0}^{\infty} s^n \\ &= (\text{constant}) + (\text{constant}) \cdot \sum_{n=1}^{\infty} s^n \\ &= (\text{constant}) + (\text{constant}) \cdot \frac{1}{1-s} < \infty \end{aligned}$$

and thus that (because its partial sums are bounded above and nondecreasing) $\sum_{n=1}^{\infty} a_n$ must converge! $\square$

So: like many of the other tests we've developed, the ratio test is far more useful than the simplicity of its proof (things that look like geometric series converge like geometric series) would indicate – we work a few examples below to show just how useful this actually is.

Example 2.2. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

converges.

Proof. So: because

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{(n)!}} = \lim_{n\to\infty} \frac{(n)!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1,$$

the ratio test tells us that this series converges. □

Example 2.3. Show that

$$\sum_{n=1}^{\infty} \frac{r^n}{n!}$$

converges, for any $r > 0$.

Proof. So: because

$$\lim_{n\to\infty} \frac{\frac{r^{n+1}}{(n+1)!}}{\frac{r^n}{(n)!}} = \lim_{n\to\infty} r \cdot \frac{(n)!}{(n+1)!} = \lim_{n\to\infty} \frac{r}{n+1} = 0 < 1,$$

the ratio test tells us that this series converges, for any value of r. In other words, we've just shown that factorials, in some tangible and real sense, grow **much** faster than powers; regardless of our choice of r – even if it was something like $10^{10^{1000}}$ – not only will the factorial eventually outgrow it, it will outgrow it fast enough to make this sum converge! □

Example 2.4. Show that

$$\sum_{n=1}^{\infty} n^k \cdot r^n$$

converges, for any $1 > r > 0$ and $k > 0$.

Proof. So: because

$$\lim_{n\to\infty} \frac{(n+1)^k r^{n+1}}{n^k r^n} = \lim_{n\to\infty} r \cdot \left(\frac{n+1}{n}\right)^k = r < 1,$$

the ratio test tells us that this series converges, for any $1 > r > 0$ and $k > 0$. So – as in the previous example – we've just shown that powers "outgrow" polynomials, in the same sense as in our example above. These useful heuristic shortcuts are often useful for evaluating sums quickly; this idea of certain kinds of terms "beating" other terms is often a useful shortcut to let you know what terms in a given series actually matter, and what sequences will be useful in applying comparison tests. □

3. The Integral Test

So: while the ratio test is remarkably powerful, it isn't the be-all and end-all of tests; specifically, while it excells at dealing with things like factorials ($n!$) or powers (2^n), it usually can't capture what's going on with polynomials or things which grow slower than polynomials. For example, while the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the limit of its terms is just

$$\lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1;$$

consequently, the ratio test tells us nothing about this series. Similarly, while $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the limit of the ratio of its terms is also

$$\lim_{n\to\infty} \frac{1/(n+1)}{1/n} = \lim_{n\to\infty} \frac{n}{(n+1)} = 1,$$

so the ratio test tells us nothing in this case as well.

So, while the ratio test is useful for many "fast-moving" series, it often fails to properly analyze series whose terms resemble reciprocals of polynomials (or similar functions.) This need for a more "sensitive" test, hopefully, should motivate our interest in the next theorem:

Theorem 3.1. *If f is a positive decreasing function on $[1, \infty)$, and $f(n) = a_n$ for every n, then the sum $\sum_{n=1}^{\infty} a_n$ converges if and only if the integral $\int_1^{\infty} f(x)dx$ exists.*

Proof. So: because $f(x)$ is decreasing, we know that for every $n \in \mathbb{N}$,

$$f(n+1) \leq f(x) \leq f(n), \qquad \forall x \in [n, n+1]$$

But this tells us that

$$\begin{aligned} & \int_n^{n+1} f(n+1)dx \leq \int_n^{n+1} f(x)dx \leq \int_n^{n+1} f(n)dx, \qquad \forall n \in \mathbb{N} \\ \Rightarrow \quad & f(n+1) \leq \int_n^{n+1} f(x)dx \leq f(n), \qquad \forall n \in \mathbb{N} \\ \Rightarrow \quad & a_{n+1} \leq \int_n^{n+1} f(x)dx \leq a_n, \qquad \forall n \in \mathbb{N} \end{aligned}$$

So: by the first convergence test, this tells us that the series $\sum_{n=1}^{\infty} a_n$ converges iff the sum $\sum_{n=1}^{\infty} \int_n^{n+1} f(x)dx$ converges; but this second sum is just the integral $\int_1^{\infty} f(x)dx$. So we're done! □

The utility of this result is probably best demonstrated by the following:

Example 3.2. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

Proof. If $p = 1$, this is just the harmonic series; so it diverges. If $p \neq 1$, by the integral test, the sum $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if

$$\int_1^{\infty} \frac{1}{x^p} dx = x^{-p+1} \cdot \frac{1}{-p+1} \Bigg|_1^{\infty} = \lim_{x\to\infty} \frac{x^{-p+1}}{-p+1} - \frac{1}{-p+1}$$

exists, which is precisely when $p > 1$. It is instructive to compare this proof with our earlier proof that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges; that proof needed a clever use of partial fractions and some ingenuity to only prove the case when $p = 2$, whereas this proof works for every single p and involved nothing more than an blind application of the integral test. So, if you ever find yourself wondering why we develop all of these tests and theorems, this hopefully illustrates some of the power that we get in exchange for working through all of these proofs. □

4. Absolute Convergence

So: pretty much every test we've developed thus far (both comparison tests, the ratio and integral tests) only work on series composed of positive terms. But we often deal with series that aren't strictly positive; for example, none of our tests can even tell us if the relatively simple series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$$

converges or diverges! This deficiency motivates the topic of this section – the notion of absolute convergence.

Definition 4.1. A series $\{a_n\}$ is called **absolutely convergent** if the sum $\sum_{n=1}^{\infty} |a_n|$ converges.

Example 4.2. The series $\left\{\frac{(-1)^{n+1}}{n}\right\}_{n=1}^{\infty}$ is not absolutely convergent, because the sum $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Conversely, the series $\left\{\frac{(-1)^{n+1}}{n^2}\right\}_{n=1}^{\infty}$ is absolutely convergent, because the sum $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

So: we have a few hopefully trivial theorems to introduce about absolute convergence, which should hopefully make clear what's going on with this notion.

Theorem 4.3. *All absolutely convergent sequences converge.*

Proof. If the sum $\sum_{n=1}^{\infty} |a_n|$ converges, then by the Cauchy criterion we have that

$$\lim_{m,n \to \infty} |a_{n+1}| + |a_{n+2}| + \ldots + |a_m| = 0$$
$$\Rightarrow \quad \lim_{m,n \to \infty} |a_{n+1} + a_{n+2} + \ldots + a_m| = 0,$$

by using the triangle inequality. But this second line is just the Cauchy criterion for the series $\sum_{n=1}^{\infty} a_n$; so this sum must converge, as well.

The use of the Cauchy criterion may seem somewhat clunky; we have generally avoided it in favor of theorems like the comparison test and similar things thus far. However, (to reiterate our earlier point) pretty much every theorem we've created thus far only works for positive sequences; so we have to revert to simpler tools until we can develop some better machinery. □

Theorem 4.4. *Given a sequence $\{a_n\}$, define the sequences $\{a_n^+\}$ and $\{a_n^-\}$ as follows:*

- $a_n^+ = a_n$ *if* $a_n > 0$, *and 0 otherwise.*
- $a_n^- = a_n$ *if* $a_n < 0$, *and 0 otherwise.*

We claim that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if and only if the sums $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ both converge.

Proof. Start by making the following observations about the $a_n^{\pm}$;

- By definition, the a_n^+'s are just the positive a_n's and the a_n^-'s are just the negative a_n's: so we have that

$$\begin{aligned} & a_n = a_n^+ + a_n^-, \forall n \\ \Rightarrow \quad & |a_n| = |a_n^+ + a_n^-| = a_n^+ - a_n^-, \forall n. \end{aligned}$$

- As well, because of the above identity, we have that

$$\begin{aligned} & |a_n| = a_n^+ - a_n^-, a_n = a_n^+ + a_n^-, \forall n \\ \Rightarrow \quad & a_n^+ = \frac{a_n + |a_n|}{2}, a_n^- = \frac{a_n - |a_n|}{2}, \forall n. \end{aligned}$$

Applying these identities to our sums in question tells us that

$$\begin{aligned} \sum_{n=1}^{\infty} |a_n| &= \left(\sum_{n=1}^{\infty} a_n^+ + \sum_{n=1}^{\infty} a_n^- \right), \\ \sum_{n=1}^{\infty} a_n^+ &= \frac{1}{2} \left(\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} |a_n| \right), \\ \sum_{n=1}^{\infty} a_n^- &= \frac{1}{2} \left(\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} |a_n| \right). \end{aligned}$$

Thus, if the sums $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ both converge, we have that the sum $\sum_{n=1}^{\infty} |a_n|$ converges, and thus that the sum $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Conversely, if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, we know from our earlier theorem that this means that $\sum_{n=1}^{\infty} a_n$ is convergent; the above equations then tell us that the sums $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ both converge, as claimed. □

So: we've shown that every sequence that is absolutely convergent. Is the opposite true? Well: consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots.$$

In our example above, we noted that this is definitely not absolutely convergent. But does it converge at all? We claim that it does, and show this in the following example:

Example 4.5.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges.

Proof. To see this, simply group the terms of this series in pairs:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \dots \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \dots \\
&= \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \dots \\
&= \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \dots \\
&= \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n)} \\
&\leq \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n-1)} \\
&= \sum_{n \text{ odd}}^{\infty} \frac{1}{n^2} \\
&\leq \sum_{n=1}^{\infty} \frac{1}{n^2}
\end{aligned}$$

which we know converges. Thus, the first comparison test tells us that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n)}$ converges; by our algebraic manipulations above, this is just $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. □

This motivates the following definition:

Definition 4.6. If a sequence converges, but does not converge absolutely, we say that it **converges conditionally**.

As we've noted before, we don't yet have much machinery for determining when a series is conditionally convergent. The following theorem will completely change that:

Theorem 4.7. *(Leibniz's Theorem.) Suppose that $\{a_n\}$ is a nonincreasing positive sequence of numbers such that*

$$\lim_{n \to \infty} a_n = 0.$$

Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof. Denote the partial sums $\sum_{k=1}^{n} a_k \cdot (-1)^{k+1}$ of our series by the symbol s_n, and consider how s_n changes as n increases. In particular, we can make three observations:

(1) $s_{2n} \leq s_{2n+2}$, for all n – i.e. the even sums form a nondecreasing sequence. This is because

$$\begin{aligned}
s_{2n+2} &= s_{2n} + a_{2n+1} \cdot (-1)^{2n+2} + a_{2n+2} \cdot (-1)^{2n+3} \\
&= s_{2n} + a_{2n+1} - a_{2n+2} \\
&\geq s_{2n}, \text{ b/c the } a_n \text{ are a nonincreasing sequence.}
\end{aligned}$$

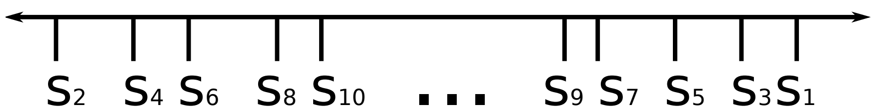
(2) $s_{2n+1} \geq s_{2n+3}$, for all n – i.e. the odd sums form a nonincreasing sequence. This is because

$$\begin{aligned} s_{2n+3} &= s_{2n+1} + a_{2n+2} \cdot (-1)^{2n+3} + a_{2n+3} \cdot (-1)^{2n+4} \\ &= s_{2n+1} - a_{2n+2} + a_{2n+3} \\ &\leq s_{2n+1}, \text{ b/c the } a_n \text{ are a nonincreasing sequence.} \end{aligned}$$

(3) $s_{2k+1} \geq s_{2l}$, for all k and l – i.e. the odd sums are always greater than the even sums. This is because for every n,

$$\begin{aligned} & s_{2n+1} = s_{2n} + a_{2n+1} \cdot (-1)^{2n+2} \\ \Rightarrow & s_{2n+1} \geq s_{2n} \\ \Rightarrow & s_{2k+1} \geq s_{2n+1} \geq s_{2n} \geq s_{2l}, \forall n > k, l, \text{ b/c of our two observations above.} \end{aligned}$$

A pictorial interpretation of these observations is in the graph of the s_n's below:



So: the picture above motivates the idea that (1) this series converges, and (2) that the value it converges to is the limit of both the odd partial sums and the even partial sums. This turns out to be true, as the following observations show:

- The odd partial sums are a nonincreasing sequence bounded below by any of the even partial sums; so they must converge! Denote the value that they converge to by x.
- Similarly, the even partial sums are a nondecreasing sequence bounded above by any of the odd partial sums; so they must converge as well. Denote the value that they converge to by y.
- Because the a_n's go to 0, we have that

$$\begin{aligned} & \lim_{n\to\infty} s_{2n} - s_{2n-1} = \lim_{n\to\infty} a_{2n} = 0 \\ \Rightarrow & \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} s_{2n-1} = 0 \\ \Rightarrow & y - x = 0 \\ \Rightarrow & x = y. \end{aligned}$$

  So these sums converge to the same thing!
- As well, we know (because the odd sums are always greater than the even sums) that

$$\begin{aligned} & \limsup_{n\to\infty} s_n = \limsup_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n+1} = x, \text{ and} \\ & \liminf_{n\to\infty} s_n = \liminf_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n} = y. \end{aligned}$$

But we just showed that $x = y$; so the limsup and the liminf of the sequence of partial sums are the same! Thus, the limit $\lim_{n\to\infty} s_n$ exists – i.e. the sum $\sum_{n=1}^{\infty} a_n$ exists. $\square$

It's worth taking a second before we go on to say a bit about how **powerful** the above theorem is: basically, if we have a sequence that (1) alternates sign and (2) its terms go to zero, then its series has to converge. This is in sharp contrast to every other test we've developed so far, where we've always been concerned with "how fast" the terms go to zero – here, we simply do not care. They can go to zero glacially slowly, so long as they head there, and we will be able to sum them up. So, while this is pretty much our only test for conditionally convergent series, we'll find that it's often more than enough for our purposes.