MATH 1D – FINAL REVIEW

INSTRUCTOR: PADRAIC BARTLETT

ABSTRACT. These are the lecture notes from the final review for Ma1d.

1. A Quick List of What We've Discussed

We've covered a massive amount of material thus far through this course; here's a list (with quick definitions and restatements) of what all exactly we've done.

(1) Sequences:
- Their definition: a sequence is just a list of numbers (or functions, or complex numbers, or whatever you wish) indexed by the natural numbers.
- When they converge: basically, we developed three methods for determining when a sequence converges:
  - The definition of convergence: a sequence $\{a_n\}$ converges to a value l if for every $\epsilon > 0$, there is a $\delta > 0$ such that $|a_n - l| < \epsilon$.
  - The squeeze/two-policeman/sandwich theorem: if $a_n \leq b_n \leq c_n$ and the sequences a_n, c_n both converge to the same limit, then b_n has no choice but to come along as well.
  - Passing to the continuous case: if f is a function such that $f(n) = a_n$ and $\lim_{x\to\infty} f(x) = l$, then $\lim_{n\to\infty} a_n = l$ also.

(2) Series:
- Their definition: an infinite series is just the sum $\sum_{n=1}^{\infty} a_n$; i.e. the limit $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$ of the partial sums of these a_n's.
- When they converge: we developed a number of tests to tell when a series converges, depending on the signs of its summands.
  If the series consists entirely of positive numbers:
  - The First Comparison Test: If $\sum a_n \leq \sum b_n$ and the sum $\sum b_n$ converges, then so does $\sum a_n$.
  - The Second Comparison Test: If $\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0$, then the sum $\sum b_n$ converges if and only if the sum $\sum a_n$ converges.
  - The Ratio Test: If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and is less than 1, the sum $\sum a_n$ converges; if it's greater than 1, the sum $\sum a_n$ diverges; if it is equal to 1 or doesn't exist, then you should try a different test.
  - The Integral Test: if f is a decreasing function such that $f(n) = a_n$, then the integral $\int_1^n f(n)$ exists iff the sum $\sum a_n$ converges.

  If the series consists of terms that alternate sign:
  - Leibniz's Theorem: if the terms a_n alternate sign and converge to 0, then their sum $\sum a_n$ converges.

  If all else fails:

   - The Cauchy Criterion: if the limit $\lim_{m,n\to\infty} |a_n + a_{n+1} + \ldots a_m| = 0$, then the sum $\sum a_n$ converges. You will almost certainly not have to use this; if you find yourself doing so, go back and try something else instead. The virtue of this test lies in that it applies to everything – functions, complex numbers, and pretty much anything you can define an absolute value for – so we often used it to prove many of our later (and more useful) tests.
- Absolute convergence: a series $\sum a_n$ converges absolutely if and only if the series $\sum |a_n|$ converges. We can change the order of summation of an absolutely convergent series to whatever we wish (a property that stands in stark contrast to the case of non-absolutely convergent series, like $\sum \frac{(-1)^n}{n}$.)

(3) Sequences and Series of Functions:
- Pointwise convergence: a sequence f_n of functions converges pointwise to a function f if and only if $\lim_{n\to\infty} f_n(x) = f(x)$, for all relevant x.
- Uniform convergence: a sequence f_n of functions converges uniformly to a function f if and only if for every $\epsilon > 0$, there is a N such that for all $n > N$, we have that $|f_n(x) - f(x)| < \epsilon$, for all relevant x. Visually, a sequence of functions f_n converges to a function f if and only if the f_n's are eventually contained within an ϵ-tube drawn around f, for any $\epsilon > 0$. This concept was more difficult, conceptually, then the idea of pointwise convergence; but the following theorems helped motivate why we cared about this:
   - Uniform convergence implies pointwise convergence; i.e. if a sequence of functions converge uniformly to a function f, then they must converge pointwise to this function as well. (note that the converse is quite false, as you all showed multiple times on HW #3.
   - Uniform convergence preserves continuity: if a sequence f_n of continuous functions converges to some function f, then f must be continuous as well.
   - Uniform convergence preserves the integral: if a sequence f_n of integrable functions converges to some function f, then $\lim_{n\to\infty} \int_a^b f_n(x) = \int_a^b f(x)$.
   - Uniform convergence kind-of preserves differentiability: if a sequence f_n of differentiable functions converges to some function f, *and* their derivatives f_n' converge uniformly to some continuous function, then $\lim_{n\to\infty} f_n' = f'$ uniformly.

(4) Power series:
- Their definition: a power series around the point c is merely a series of functions of the form $\sum a_n(x-c)^n$.
- Massively Useful Theorem: If a power series $\sum a_n x^n$ converges at some point x_0, then:
   - $\sum a_n x^n$ converges uniformly on the interval $[-a, a]$, for any $a < |x_0|$,
   - the series $\sum n a_n x^{n-1}$ converges uniformly on the interval $[-a, a]$ as well, and
   - $\frac{d}{dx}(\sum a_n x^n) = \sum n a_n x^{n-1}$.

So, in other words, we have the following: if $f(x) = \sum a_n x^n$ is a power series convergent over some region, then

- $f'(x) = \sum n a_n x^{n-1}$, and
- $\int f(x)dx = \sum \frac{a_n x^{n+1}}{n+1}$, up to a constant C.

(5) Complex Power Series:

- First, recall the basic properties of the complex numbers, which are quickly gone over in a list in week 6. Basically, they're what you get when you throw in $\sqrt{-1}$ into the real numbers, you can factor all polynomials into roots, and you can write any complex number in the form $re^{i\theta}$, where r is the distance of the number from the origin and θ is angle made by the line $\overline{0,z}$ and the positive x-axis.
- A complex power series, then, is just a power series $\sum a_n(z-c)^n$, where the a_n's and c are complex numbers.
- Then, basically, everything that was true about real power series follows for complex power series! In particular, we have (again) that if $f(z) = \sum a_n z^n$ is a complex power series convergent over some region, then
  - $f'(z) = \sum n a_n z^{n-1}$, and
  - $\int f(z)dx = \sum \frac{a_n z^{n+1}}{n+1}$, up to a constant C.
- Motivated by these concepts, we decided to use power series to define the functions

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \ldots,$$
$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \ldots, \text{and}$$
$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \ldots,$$

  and saw that Euler's formula,

$$e^z = \cos(z) + i\sin(z)$$

  holds.
- Finally, we finished things up by using all of this machinery (plus the Weierstrass factorization theorem, a very large *deus ex machina* we discussed on Thursday, wk. 6 – you definitely don't need to use this!) to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

So: up through week 6, that's everything we've covered in this course. (The last two lectures aren't necessary for the final, though Tuesday, wk. 7's lecture will make problem 6 far easier than it is without attending.)

2. Example Problems

To help illuminate some of the above ideas, here are five example problems that were shortlisted for your final, one from each section:

Example 2.1. Let

$$a_n := \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \log(n).$$

Does the sequence of a_n's converge?

Proof. So: first, note that because

$$\frac{1}{n+1} \leq \int_n^{n+1} \frac{1}{x} dx \leq \frac{1}{n},$$

we have

$$(\star) \qquad \frac{1}{n+1} \leq \log(n+1) - \log(n) \leq \frac{1}{n},$$

for every $n \geq 1$.

So: why do we care? Well, look at the elements a_n. We can write

$$\begin{aligned}
a_{n+1} &= \left(\sum_{k=1}^{n+1} \frac{1}{k}\right) - \log(n+1) \\
&= \left(\sum_{k=1}^{n} \frac{1}{k}\right) + \frac{1}{n+1} - \log(n+1) \\
&= \left(\sum_{k=1}^{n} \frac{1}{k}\right) - (\log(n) - \log(n)) + \frac{1}{n+1} - \log(n+1) \\
&= \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \log(n) - \left(\log(n+1) - \log(n) - \frac{1}{n+1}\right).
\end{aligned}$$

But, by $(\star)$, we know that $\left(\log(n+1) - \log(n) - \frac{1}{n+1}\right)$ is greater than 0, for every $n \geq 1$; so the a_n's form a nonincreasing sequence. As well, because $\int_n^{n+1} \frac{1}{x} dx \leq \frac{1}{n}$ holds for every n, we have that

$$\begin{aligned}
& \sum_{n=1}^{N} \int_n^{n+1} \frac{1}{x} dx \leq \sum_{n=1}^{N} \frac{1}{n} \\
\Rightarrow \quad & \int_1^{N+1} \frac{1}{x} dx \leq \sum_{n=1}^{N} \frac{1}{n} \\
\Rightarrow \quad & \log(N+1) \leq \sum_{n=1}^{N} \frac{1}{n} \\
\Rightarrow \quad & 0 \leq \sum_{n=1}^{N} \frac{1}{n} - \log(N+1) \\
\Rightarrow \quad & 0 \leq \sum_{n=1}^{N+1} \frac{1}{n} - \log(N+1);
\end{aligned}$$

so all of these a_n's are positive. So they form a nondecreasing positive sequence; so they converge!

For more information on the number that this sequence converges to, check out the Wikipedia entry for the Euler-Mascheroni constant, which has a lot of cool information. Amongst other things, it's still an open question in mathematics to decide if this constant is rational! Cool stuff. □

Example 2.2. Does the series

$$\sum_{n=2}^{\infty} \frac{1}{(\log(n))^{\log(n)}}$$

converge?

Proof. Lacking any better ideas, we'll use the integral test, and thus try to show that

$$\int_2^{\infty} \frac{1}{(\log(x))^{\log(x)}} dx$$

exists. This looks horrible; so what can we do? Well; perhaps we can integrate by substitution? The only immediately sane choice here is $u = \log(x)$; in this situation, we have that $du = \frac{1}{x} dx$, or in other words $du = \frac{1}{e^u} dx$; i.e $e^u du = dx$. Substiution now yields

$$\int_{\log(2)}^{\infty} \frac{e^u}{u^u} du;$$

which, at first glance, doesn't seem much better. So how can we tackle this? Well – how about the integral test again? (Madness; I know.) Specifically, the integral test tells us that this integral converges iff the sum

$$\sum_{n=1}^{\infty} \left(\frac{e}{n}\right)^n$$

converges. But for all $n \geq 3$, we have that $\left(\frac{e}{n}\right)^n \leq \left(\frac{e}{3}\right)^n$. Thus, because the series

$$\sum_{n=1}^{\infty} \left(\frac{e}{3}\right)^n$$

is geometric, it forces the series $\sum_{n=1}^{\infty} \left(\frac{e}{n}\right)^n$ to converge by the comparison test; so by our two above uses of the integral test, we then have that $\int_{\log(2)}^{\infty} \frac{e^u}{u^u} du$ and $\sum_{n=2}^{\infty} \frac{1}{(\log(n))^{\log(n)}}$ exist! So we're done. □

Example 2.3. Find a sequence of piecewise continuous functions f_n that converge pointwise to the function

$$f(x) := \begin{cases} 1, & \text{if } x \notin \mathbb{Q} \text{ and } x \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Can any such sequence converge uniformly to f?

Proof. So: take

$$f_n(x) := \begin{cases} 0, & x \text{ of the form } \frac{p}{m}, \text{ where } m \leq n \text{ and } p \in \mathbb{Z} \\ 1, & \text{if } x \text{ is in } [0,1] \text{ and not of the form above,} \\ 0, & x \notin [0,1]. \end{cases}$$

These functions all have only finitely many discontinuities; so they are piecewise continuous. But their pointwise limit is clearly f, as $f_n(\frac{p}{q})$ is 1 for any $n \geq q$, and $f_n(x) = 0$ for any irrational x.

As for uniform convergence: we claim that no such sequence of functions can hope to converge uniformly to f. To see why, recall the definition of uniform convergence: it says that for any $\epsilon > 0$ there is a n such that $|f_n(x) - f(x)| < \epsilon$ for all x. So, choose $\epsilon = \frac{1}{8}$ here, say; then the functions f_n have to be within $1/8$ of 0 at every rational point in $[0, 1]$ and within $1/8$ of 1 at every irrational point in $[0, 1]$, for sufficiently large n. But this means that the f_n's can be continuous nowhere in $[0, 1]$, as in any neighborhood of any point $x \in [0, 1]$, they are both greater than $7/8$ and less than $1/8$. In particular, the f_n cannot be piecewise continuous functions; so we're done! □

Example 2.4. Find the sum

$$\frac{x^2}{2} - \frac{x^3}{3 \cdot 2} + \frac{x^4}{4 \cdot 3} - \frac{x^5}{5 \cdot 4} + \ldots,$$

for any $x \in (-1, 1)$.

Proof. So: notice first that the derivative of the above is just

$$2\frac{x}{2} - 3\frac{x^2}{3 \cdot 2} + 4\frac{x^3}{4 \cdot 3} - 5\frac{x^4}{5 \cdot 4} + \ldots = \qquad x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots,$$

which is simply the Taylor series for $\log(1 + x)$ in the interval $(-1, 1)$.

So: this tells us that our above power series is just

$$\begin{aligned} \int \log(1+x)dx &= \int \log(u)du \\ &= u\log(u) - u + C \\ &= (x+1)\log(x+1) - (x+1) + C. \end{aligned}$$

To solve for C, just plug in $x = 0$; as the above power series is identically 0 at 0, we then have that

$$\begin{aligned} & (0+1)\log(0+1) - (0+1) + C = 0 \\ \Rightarrow \quad & C = 1, \end{aligned}$$

and thus that

$$(x+1)\log(x+1) - x = \frac{x^2}{2} - \frac{x^3}{3 \cdot 2} + \frac{x^4}{4 \cdot 3} - \frac{x^5}{5 \cdot 4} + \ldots.$$

So we're done! □

Example 2.5. Consider the three sums

$$(A) := \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \qquad (B) := \sum_{n=1}^{\infty} \frac{z^n}{n}, \qquad (C) := \sum_{n=1}^{\infty} z^n.$$

Show that (A) converges for every point z on the unit circle, (B) converges for some points on the unit circle, but not for all such points, and (C) diverges at any point on the unit circle.

Proof. So: first note that any point z on the unit circle, because its distance from the origin is 1, satisfies $|z| = 1$, and thus can be written in the form $z = e^{i\theta}$, for some angle θ.

Given this: examine (A) first, and try using the Cauchy criterion. If $|z| = 1$, we have that the limit

$$\lim_{m,n\to\infty} \left| \frac{z^m}{m^2} + \frac{z^{m+1}}{(m+1)^2} + \ldots + \frac{z^n}{n^2} \right| \leq \lim_{m,n\to\infty} \left| \frac{z^m}{m^2} \right| + \left| \frac{z^{m+1}}{(m+1)^2} \right| \ldots + \left| \frac{z^n}{n^2} \right|$$
$$\leq \sum_{n=m}^{\infty} \frac{|z^n|}{n^2}$$
$$= \sum_{n=m}^{\infty} \frac{1}{n^2},$$

which goes to 0 as m goes to infinty (as $\sum \frac{1}{n^2}$ converges.) Thus, the Cauchy criterion tells us that this series converges for any z on the unit circle!

For (B): trivially, letting $z = 1$ makes it so that this diverges, as it's just the harmonic series; similarly, letting $z = -1$ makes the series converge, as it's just the alternating harmonic series, which converges by Leibniz.

For (C): take any z with $|z| = 1$. Then the terms z^n all also have magnitude 1, and thus $\lim_{n\to\infty} z^n$ is not equal to zero! But this means that this sum cannot possibly converge (as its terms never "settle down" as n grows large.) □