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Recitation 6: Integration in $\mathbb{R}^n$

Week 6 *Caltech 2013*

1 Integration in $\mathbb{R}^n$

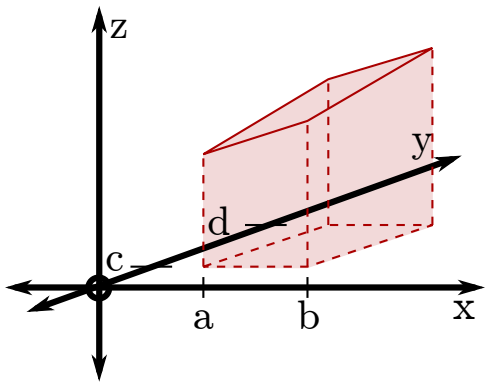
In the first five weeks of this course, we've introduced topics like limits, derivatives and optimization for functions on $\mathbb{R}^n$; furthermore, whenever we've done so, we've built all of our understanding and tools by looking at the 1-dimensional case, and extending our knowledge of functions on $\mathbb{R}^1$ to the study of $\mathbb{R}^n$. Today's lecture, on integration in $\mathbb{R}^n$, will be another class in this format!

In $\mathbb{R}^1$, we had two ways of looking at the definite integral of a function $f(x)$, $\int_a^b f(x)dx$. One was thinking of the integral as the **area under the curve** of $f(x)$ from a to b: in other words, the area of the region bounded by the lines $x = a, x = b, y = 0$ and the curve $f(x) = y$. Another, which we discussed a bit less, was the idea of the integral as the **average of** $f(x)$ from a to b, multiplied by the length of the interval $[a, b]$.

How can we "generalize" the integral to something we can calculate for functions $f : \mathbb{R}^n \to \mathbb{R}$? Well: first, we should generalize the concept of an interval $[a, b]$ to a box in $\mathbb{R}^n$: i.e. a region of the form $[a, b] \times [c, d] \times \dots$ in $\mathbb{R}^n$. If we've done this, then the natural generalization of our "area" concept, at least for functions $\mathbb{R}^2 \to \mathbb{R}$, is the idea of **volume**: i.e. we can define the integral of $f(x, y)$ over some box $[a, b] \times [c, d]$,

$$\int_{[a,b]\times[c,d]} f(x, y)dA,$$

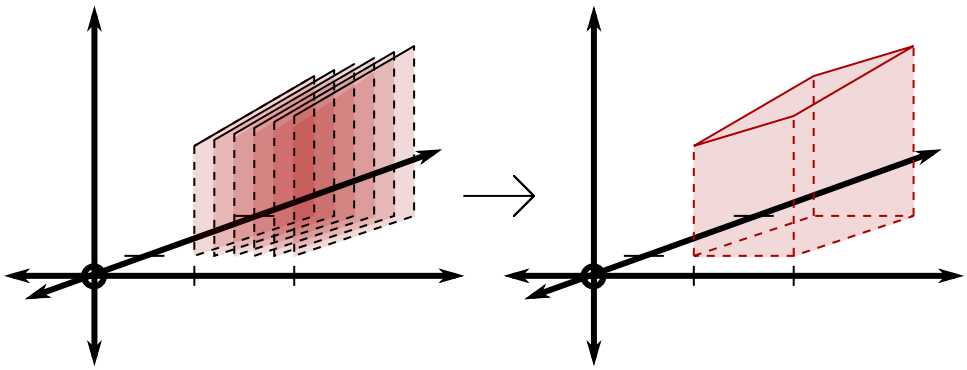
as simply the volume of the region bounded below by the plane $z = 0$, above by the surface $f(x, y) = z$, and with x, y coördinates constrained to the box $[a, b] \times [c, d]$. (The dA in the expression above is a reminder that we're integrating over a 2-dimensional region, and therefore that the "tiny bits" that we're using to integrate f are 2-dimensional, as opposed to one-dimensional like dx or dy.)



Similarly, if we'd rather extend the idea of "averages," we can think of the integral of a function in $f : \mathbb{R}^n \to \mathbb{R}$ over some box B as the average value of f over this entire box B, multiplied by the volume of the box B.

We now have a way to think about the integral in $\mathbb{R}^n$! The next natural thing to want to study, then, is how we'd ever actually go about *calculating* an integral: specifically, how we can use our past knowledge of integration in $\mathbb{R}^1$ to calculate integrals in $\mathbb{R}^n$. For the moment, let's think about functions $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$. If we're thinking of the integral in the above "volume" sense, then the following process makes sense as a possible way to calculate the integral in $\mathbb{R}^2$:

- Start with a function $f(x, y)$. We want to find the volume of the region between $z = f(x, y)$ and the plane $z = 0$, restricted to the box $[a, b] \times [c, d]$.
- To do this, pick any constant value $\lambda \in [a, b]$, and calculate the one-dimensional integral $\int_c^d f(\lambda, y)dy$. This is giving you the "area" of various cross-sections of the region we're studying, corresponding to the slices we get by setting x equal to said constant.
- Now, to combine all of these areas into a volume, simply integrate the function that spits out all of these areas, $\int_c^d f(x, y)dy$ (a function in one variable, x) over the interval $[c, d]$. The result is the average of these areas over the interval $[c, d]$, times the length of $[c, d]$: i.e. it's the **volume**!



Of course, we can also slice our surface by setting y constant first, and then integrating with respect to y: the picture will be the same. The upshot of all of this is that we can now calculate integrals in $\mathbb{R}^2$ using only integrals in $\mathbb{R}^1$: i.e. that

$$\int_{[a,b]\times[c,d]} f(x, y)dA = \int_a^b \left(\int_c^d f(x, y)dy \right) dx = \int_c^d \left(\int_a^b f(x, y)dx \right) dy.$$

The above concept of the integral as an "average" will give you the exact same conclusion, as well. I generally prefer thinking of the integral as an average $\times$ the area of the thing we're integrating over, if only because it makes certain generalizations easier: i.e. using this concept, it's really easy to see that our above discussion for $\mathbb{R}^2$ generalizes completely to $\mathbb{R}^n$. For example, in $\mathbb{R}^3$, we can see that

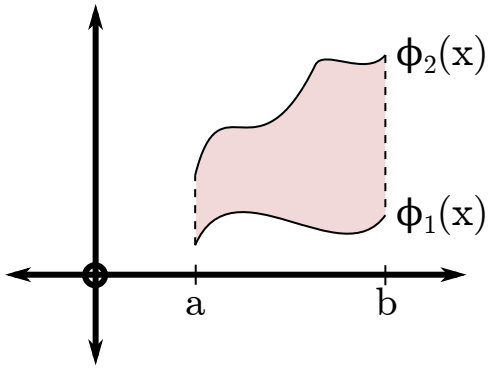
$$\int_{[a,b]\times[c,d]\times[e,g]} f(x, y, z)dA = \int_a^b \left(\int_c^d \left(\int_e^g f(x, y, z)dz \right) dy \right) dx.$$

2 x-Simple Regions

As well, if we want to integrate over regions that aren't boxes, this method generalizes beautifully! Let's think about functions $\mathbb{R}^2 \to \mathbb{R}$ for the moment. Suppose that we have a region D of the form

$$D = \{(x, y) : x \in [a, b], y \in [\phi_1(x), \phi_2(x)]\};$$

i.e. D looks like



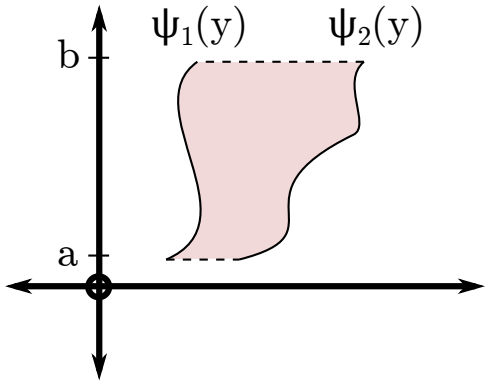
Then, if we want to calculate the integral of $f(x, y)$ over this region D, we'd want to perform the same slicing-into-areas trick as before: i.e. we would want to find the integrals of $f(\lambda, y)$ over $[\phi_1(\lambda), \phi_2(\lambda)]$ for every λ, and then we'd want to combine these areas into a volume by integrating with respect to x. In other words, we have

$$\int_D f(x, y)dA = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy \right) dx.$$

We call such regions **x-simple**, because they are regions that we can define by constraining x simply to an interval and then deriving curves that bound our choices of y. Similarly, a **y-simple** region D is one of the form

$$D = \{(x, y) : y \in [c, d], x \in [\psi_1(y), \psi_2(y)]\};$$

i.e. where D looks like



We can calculate integrals over a y-simple region just like we did for x-simple regions:

$$\int_D f(x,y)dA = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right) dy.$$

Sometimes, we can express a region D as both a x-simple and y-simple region! I.e. the upper-left quadrant of the unit circle can be expressed as both

$$\{(x,y) : x \in [0,1], y \in [0, \sqrt{1-x^2})]\}, \text{ and } \{(x,y) : y \in [0,1], x \in [0, \sqrt{1-y^2}]\}.$$

In regions where we can pull off this "two different descriptions" trick, we can often use these two different descriptions to study integrals that would be otherwise impossible.

3 Change of Variables

Finally, to describe regions that are even more complicated than x-simple or y-simple regions, we have the technique of **change of variables.** The concept of changing variables is one we've ran into in single-variable calculus:

Theorem 1 *(Change of variables, single-variable form) Suppose that f is a continuous function over the interval $(g(a), g(b))$, and that g is a $1-1$ continuous map with continuous derivative from (a,b) to $(g(a), g(b))$. Then, we have that*

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x)) \cdot g'(x)dx.$$

The idea here, roughly, was the following: the integral of f over the interval $(g(a), g(b))$ is the same as the integral of $f \circ g$ over the interval (a,b), as long as we correct for how g "distorts space." In other words, on the left (where g's been applied to the domain (a,b)), we're integrating with respect to dx, the change in x: however, on the right, we're now integrating $f \circ g$, and therefore we should integrate with respect to $d(g(x)) = g'(x)dx$.

So: in multiple variables, we want to have a similar theorem! Basically, given a continuous function $f : \mathbb{R}^n \to \mathbb{R}$, a domain $R \subset \mathbb{R}^n$, and a differentiable map $g : \mathbb{R}^n \to \mathbb{R}^n$, we want a way to relate the integral of f over $g(R)$ and the integral of $f \circ g$ over R.

How can we do this? In other words, how can we correct for how g "distorts space," like we did for our single-variable case? Well: locally, we know that small changes in the vector $\mathbf{x}$ are measured by $D(g(\mathbf{x}))$, the $n \times n$ matrix of partial derivatives of g. Specifically, from Math 1b, we know that $\det(D(g(\mathbf{x})))$ measures the volume of the image of the unit cube under the map $D(g(\mathbf{x}))$. So, in a sense, this quantity – $\det(D(g(\mathbf{x})))$, the determinant of the Jacobian of g – is telling us how much g is locally inflating or shrinking space at the point $\mathbf{x}$! So, we might hope that this is the correct quantity to scale by. As it turns out, it is! Specifically, we have the following theorem:

Theorem 2 *(Change of variables, multiple-variable form) Suppose that R is a region in $\mathbb{R}^n$, g is a $1-1$ map with continuous partial derivatives that maps R to some region $g(R) \subset \mathbb{R}^n$, and that f is a continuous function. Then, we have*

$$\int_{g(R)} f(\mathbf{x})dV = \int_R f(g(\mathbf{x})) \cdot \det(Dg(\mathbf{x}))dV.$$

This, as you may have noticed, is not precisely the theorem in your textbook. That theorem reads as follows:

Theorem 3 *(Change of variables, multiple-variable form) Suppose that R is a region in $\mathbb{R}^n$, g is a 1-1 map with continuous partial derivatives that maps $g^{-1}(R)$ to R, and that f is a continuous function. Then, we have*

$$\int_R f(\mathbf{x})dV = \int_{g^{-1}(R)} f(g(\mathbf{x})) \cdot \det(Dg(\mathbf{x}))dV.$$

The change between the two theorems lies in how you're thinking of the integral you've started with: you can consider it either as a integral where you've already got a good guess for what your g-map will be ($\int_{g(R)} f(\mathbf{x})dV$), or you can think of it in the situation where you don't have an idea what your g-map is yet ($\int_R f(\mathbf{x})dV$).

If you use either of these results, you need to be very very careful to insure that your map g is 1-1 on your region R! Otherwise, it's possible that g^{-1} will "fold" parts of R on top of each other, in such a way that the integral at the right will no longer be over something that looks like R. For example, if your map g was the map $(x, y) \mapsto (x^2, y^2)$, and you were to try applying this map to the integral

$$\int_{g([-1,1]\times[0,1])} 1 \; dV,$$

you'd get

$$\int_{g([-1,1]\times[0,1])} 1 \; dV =^{??} \int_{[-1,1]\times[0,1]} 1 \cdot \det \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix} dV = \int_{-1}^{1}\int_{0}^{1} (4xy)dydx = 2,$$

even though

$$\int_{g([-1,1]\times[0,1])} 1 \; dV = \int_{[0,1]^2} 1 \; dV = \int_0^1 \int_0^1 dxdy = 1.$$

This is because the map g "folds" the region $[-1, 1] \times [0, 1]$ we were integrating over in half: therefore, if we use change of variables to change the region we're integrating over from $g([-1, 1] \times [0, 1])$ to $[-1, 1] \times [0, 1]$, we'd expect to see an "unfolding", which would cause our integral to double. (Which is precisely what we saw!)

4 Common variable changes.

There are three very common changes of variable, which we review here briefly:

Theorem 4 *(Change of variables, polar:) Let $\gamma : [0, \infty) \times [0, 2\pi)$ be the polar coördinates map $(r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$. Then $D(\gamma(r, \theta)) = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$, $\det(D(\gamma(r, \theta))) = r$, and we have*

$$\int_{\gamma(R)} f(x, y)dV = \int_R f(r\cos(\theta), r\sin(\theta)) \cdot rdV,$$

for any region $R \subset [0, \infty) \times [0, 2\pi)$, and any continuous function f on an open neigborhood of R.

In other words, if we have a region R described by polar coördinates, we can say that the integral of f over $\gamma(R)$ is just the integral of $r \cdot f(r\cos(\theta), r\sin(\theta))$ over this region interpreted in Euclidean coördinates. For example, suppose that R was the unit disk, which we can express using our polar coördinates map as $\gamma([0,1] \times [0, 2\pi))$. Then, change of variables tells us that the integral of f over the unit disk is just the integral of $r \cdot f(r\cos(\theta), r\sin(\theta))$ over the Euclidean-coördinates rectangle $[0,1] \times [0, 2\pi)$.

Cylindrical coördinates are similar:

Theorem 5 *(Change of variables, cylindrical:) Let $\gamma : [0,\infty) \times [0, 2\pi) \times \mathbb{R}$ be the cylindrical coördinates map $(r, \theta, z) \mapsto (r\cos(\theta), r\sin(\theta), z)$. Then $D(\gamma(r,\theta)) = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\det(D(\gamma(r,\theta))) = r$, and we have*

$$\int_{\gamma(R)} f(x,y)dV = \int_R f(r\cos(\theta), r\sin(\theta), z) \cdot r dV,$$

for any region $R \subset [0,\infty) \times [0, 2\pi) \times (-\infty, \infty)$, and any continuous function f on an open neigborhood of R.

Spherical coördinates are a bit trickier, but have a similar form:

Theorem 6 *(Change of variables, spherical:) Let $\gamma : [0,\infty) \times [0, \pi) \times [0, 2\pi)$ be the cylindrical coördinates map $(r, \varphi, \theta) \mapsto (r\cos(\varphi), r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta))$. Then $D(\gamma(r,\theta)) = \begin{bmatrix} \cos(\varphi) & -r\sin(\varphi) & 0 \\ \sin(\varphi)\cos(\theta) & r\cos(\varphi)\cos(\theta) & -r\sin(\varphi)\sin(\theta) \\ \sin(\varphi)\sin(\theta) & r\cos(\varphi)\sin(\theta) & r\sin(\varphi)\cos(\theta) \end{bmatrix}$, $\det(D(\gamma(r,\theta))) = r^2\sin(\varphi)$, and we have*

$$\int_{\gamma(R)} f(x,y)dV = \int_R f(r\cos(\varphi), r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta)) \cdot r^2\sin(\varphi) dV,$$

for any region $R \subset [0,\infty) \times [0, \pi) \times [0, 2\pi)$ and any continuous function f on an open neigborhood of R.

There are a few other coördinate transforms that will often come up:

- Various translations of space: i.e. maps $(x, y, z) \mapsto (x + c_1, y + c_2, z + c_3)$. The determinant of the Jacobian of such maps will always be 1.
- Various ways to scale space: i.e. maps $(x, y) \mapsto (\lambda_1 x, \lambda_2 y)$. The determinant of the Jacobian of such maps will be the product of these scaling constants $\lambda_1 \cdot \ldots \cdot \lambda_n$.
- Various compositions of these maps: i.e. a translation map, followed by a spherical coördinates map, followed by a scaling map. Using the chain rule, the determinant of the Jacobian of any such composition of maps is just the product of the determinants of the individual Jacobians.

5 Examples

Next week, we'll have many more examples of these techniques. For now, however, here's three worked examples to give you an idea of how integrals work in $\mathbb{R}^n$.

Question 7 *Choose two random numbers from* $[0,1]$. *What is the average value of the smaller of the two? In other words, what is the integral (average value)*

$$\frac{1}{Area\,([0,1]^2)} \cdot \iint\limits_{[0,1]\times[0,1]} \min(x,y)dA \quad ?$$

Solution. First: notice that the area of $[0,1]^2$ is just 1, so $\frac{1}{\text{Area}([0,1]^2)}$ is just 1. So we can just study the integral!

Using our discussion above, start by expressing this integral as two nested one-dimensional integrals:

$$\iint\limits_{[0,1]\times[0,1]} \min(x,y)dA = \int_0^1 \left(\int_0^1 \min(x,y)dx \right) dy.$$

With this done, let's study the inner integral. Directly working with the function $\min(x,y)$ seems difficult. However, for a given fixed value of y, notice that we can split our integral into two parts (the integral from 0 to y and the integral from y to 1):

$$\int_0^1 \min(x,y)dx = \int_0^y \min(x,y)dx + \int_y^1 \min(x,y)dx$$

Then, if we notice that $\min(x,y)$ is just x whenever $x \leq y$ (i.e. x is in our first part) and is just y whenever $x \geq y$ (i.e. x is in our second part), we can replace the complicated $\min(x,y)$'s with just x and y: i.e.

$$\begin{aligned}
\int_0^1 \min(x,y)dx &= \int_0^y \min(x,y)dx + \int_y^1 \min(x,y)dx \\
&= \int_0^y x dx + \int_y^1 y dx \\
&= \frac{x^2}{2}\Big|_0^y + xy\Big|_y^1 \\
&= \frac{y^2}{2} + y - y^2 \\
&= y - \frac{y^2}{2}.
\end{aligned}$$

Therefore, if we plug this into our nested integrals, we have that

$$\begin{aligned}\iint_{[0,1]\times[0,1]} \min(x,y)dA &= \int_0^1 \int_0^1 \min(x,y)dxdy \\ &= \int_0^1 \left(y - \frac{y^2}{2}\right) dy \\ &= \frac{y^2}{2} - \frac{y^3}{6}\Big|_0^1 \\ &= \frac{1}{3}.\end{aligned}$$

So, if you take two random numbers in $[0,1]$ and look at the smaller of the two, the average value you'd see is $1/3$.

Question 8 *Now, choose two random positive numbers so that their sum is between 0 and 1. What is the average value of the smaller of the two?*

Solution. Similarly to before, we want to integrate $\min(x,y)$ over some region: however, our region is now

$$D = \{(x,y) : 0 \le x, 0 \le y, x+y \le 1\} = \{(x,y) : y \in [0,1], x \in [0,1-y]\}.$$

We can use this description of D to notice that it's a right triangle, with sides given by the lines $x = 0, y = 0, x + y = 1$. So its area is just $\frac{1}{2}$, so $\frac{1}{\text{Area}(D)} = 2$.

Now, again using this description of D, we can express our integral as the following expression:

$$2 \cdot \iint_{[0,1]\times[0,1]} \min(x,y)dA = \int_0^1 \left(\int_0^{1-y} \min(x,y)dx\right) dy.$$

With this done, let's study the inner integral. It's now a bit more complicated! In specific, notice that if we have $y \ge \frac{1}{2}$, because $x \in [0,1-y]$, we will always have $x < y$: i.e. $\min(x,y) = x$. So, for $y \ge \frac{1}{2}$, we have

$$\int_0^{1-y} \min(x,y)dx = \int_0^{1-y} xdx = \frac{x^2}{2}\Big|_0^{1-y} = \frac{(1-y)^2}{2} = \frac{y^2-2y+1}{2}.$$

However, if $y \le \frac{1}{2}$, it's possible that $x > y$ or $x < y$, and we are led to perform the same trick as before of splitting our integral into two pieces, one from 0 to y and the other from

y to $1-y$:

$$\begin{aligned}\int_0^{1-y} \min(x,y)dx &= \int_0^y \min(x,y)dx + \int_y^{1-y} \min(x,y)dx \\ &= \int_0^y x dx + \int_y^{1-y} y dx \\ &= \frac{x^2}{2}\Big|_0^y + xy\Big|_y^{1-y} \\ &= \frac{y^2}{2} + (y - y^2) - y^2 \\ &= y - \frac{3y^2}{2}.\end{aligned}$$

Therefore, if we plug these results into our nested integrals, we get that

$$\begin{aligned}2 \cdot \iint_{[0,1]\times[0,1]} \min(x,y)dA &= 2 \cdot \int_0^1 \int_0^1 \min(x,y)dxdy \\ &= 2 \cdot \left(\int_0^{1/2} \left(y - \frac{3y^2}{2} \right) dy + \int_{1/2}^1 \left(\frac{y^2 - 2y + 1}{2} \right) dy \right) \\ &= 2 \cdot \left(\left(\frac{y^2}{2} - \frac{y^3}{2} \right) \Big|_0^{1/2} + \left(\frac{y^3}{6} - \frac{y^2}{2} + \frac{y}{2} \right) \Big|_{1/2}^1 \right) \\ &= 2 \cdot \left(\left(\frac{1}{8} - \frac{1}{16} \right) - 0 + \left(\frac{1}{6} - \frac{1}{2} + \frac{1}{2} \right) - \left(\frac{1}{48} - \frac{1}{8} + \frac{1}{4} \right) \right) \\ &= \frac{1}{6}\end{aligned}$$

Therefore, if you take two random positive numbers such that their sum is ≤ 1 and you look at the smaller of the two, you'll get $\frac{1}{6}$ on average.

Question 9 *Choose a random point in the upper-right quadrant of the unit disk: i.e. a random point (x, y) such that $x, y \geq 0, x^2 + y^2 \leq 1$. What is the average value of the minimum of (x, y)?*

Solution. We are looking for the average value of the function $\min(x, y)$ over this upper-right quadrant of the unit disk, i.e.

$$\frac{1}{\text{area(part of unit disk)}} \int_{\text{(part of unit disk)}} \min(x,y)dA.$$

Polar coördinates look like a good candidate for how we can find this integral! In particular, if we apply the change of variables formula using the polar coördinate transform

$g(r,\theta) = (r\cos(\theta), r\sin(\theta))$ to the above integral, we get

$$\begin{aligned}\int_{(\text{part of unit disk})} \min(x,y)dA &= \int_0^1 \int_0^{\pi/2} \min(r\cos(\theta), r\sin(\theta)) \cdot \det(Dg) \cdot d\theta dr \\ &= \int_0^1 \int_0^{\pi/2} \min(r\cos(\theta), r\sin(\theta)) \cdot r \cdot d\theta dr.\end{aligned}$$

Note that we can do because the polar-coördinate transform is $1-1$ on the set $[0,1]\times[0,\pi/2]$, which is precisely the set that maps onto the upper-right quadrant of the unit disk.

With this done, we can simply integrate. First, notice that from 0 to $\pi/4$, $\sin(\theta) \leq \cos(\theta)$, and from $\pi/4$ to π, we have that $\cos(\theta) \leq \sin(\theta)$. Therefore, if we break our inner integral into two parts, one going from 0 to $\pi/4$ and the other from $\pi/4$ to $\pi/2$, we have

$$\begin{aligned}&\int_0^1 \int_0^{\pi/2} r\min(r\cos(\theta), r\sin(\theta))\ d\theta dr \\ =&\int_0^1 \left(\int_0^{\pi/4} r\min(r\cos(\theta), r\sin(\theta)) d\theta + \int_{\pi/4}^{\pi/2} r\min(r\cos(\theta), r\sin(\theta)) d\theta \right) dr \\ =&\int_0^1 \left(\int_0^{\pi/4} r^2\sin(\theta) d\theta + \int_{\pi/4}^{\pi/2} r^2\cos(\theta) d\theta \right) dr \\ =&\int_0^1 \left(r^2(-\cos(\theta))\Big|_0^{\pi/4} + r^2(\sin(\theta))\Big|_{\pi/4}^{\pi/2} \right) dr \\ &= \int_0^1 r^2 \left(1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + 1 \right) dr \\ &= (2-\sqrt{2})\frac{r^3}{3}\Big|_0^1 \\ &= \frac{2-\sqrt{2}}{3}.\end{aligned}$$

Therefore, if we want the average value of $\min(x,y)$ over this upper-right quadrant, we just have to divide our integral by the area of the upper-right quadrant, $\pi/4$. In other words, we've just proven that the average value of $\min(x,y)$ is

$$\frac{8-4\sqrt{2}}{3\pi},$$

which is roughly $\frac{1}{4}$.