Math 1c	TA: Padraic Bartlett
Recitation 4: Arc Length; Div, Grad, and Curl	
Week 4	*Caltech 2013*

Our recitation today is centered around two fairly different things: the formula for arc length, and the concepts of divergence, gradients, and curl. We study each separately below.

1 Arc Length

We open with some theoretical discussion for how you might go about deriving the formula for arc length. If this isn't what you care about, feel free to skip forward to the part where we actually use this formula in a page or so.

Recall, from earlier in class, the definition of a **path** in $\mathbb{R}^n$:

Definition. A **path** is simply a function $\gamma : [a, b] \to \mathbb{R}^n$, from some interval $[a, b]$ to $\mathbb{R}^n$.

The graph of a path over its entire input interval $[a, b]$ is some sort of curve in n-dimensional space. A natural question to ask, given a path, is what the **length** of this path is in $\mathbb{R}^n$!

A particularly useless answer to this question is the following integral:

$$\text{arc\_length}(\gamma) = \int_\gamma ds,$$

where ds denotes the signed length of "tiny" portions of the curve we're integrating over (formally, the "differential" formed by infinitesimally small pieces of our curve), and where our region of integration is precisely the graph of γ. In other words, if we add up the length of tiny pieces of the curve over the entire curve, we should, um, get the length of the whole curve.

This formula, like many things in mathematics without proper context, is completely true but also completely useless. I.e. what we want to know is how to actually **perform** such a calculation – i.e. how to turn this integral into something we can actually find, and that isn't just symbolic nonsense.

This, however, we **can** do! – and furthermore, can do by examining our definitions, and thereby turn this symbolic expression into something that's actually useful. Specifically: we want to take our integral over γ of these little ds-bits, and turn it into something over $[a, b]$ (because we understand how to integrate over intervals in $\mathbb{R}$ — this is just single-variable calculus!) To do this, we just need to express these little bits of length ds as something with respect to dt (where dt denotes a little bit of change in the inputs from $[a, b]$ to $\gamma(t)$.

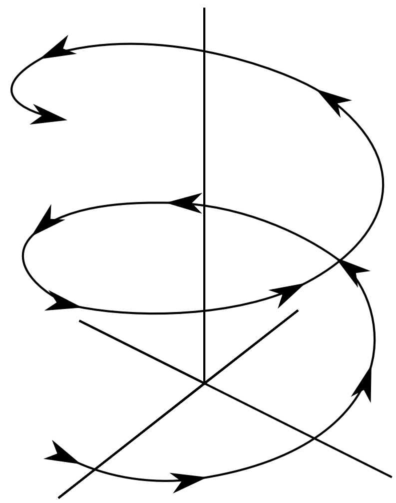
So: a little bit of change in t produces a little bit of change $\frac{d\gamma_1}{dt}dt$ in the x_1-coordinate, and a little bit of change $\frac{d\gamma_2}{dt}dt$ in the x_2-coordinate, and more generally a little bit of change $\frac{d\gamma_k}{dt}$ in every coordinate x_k of our space. Therefore, because we can express the little changes in our curve, ds, as just the square root of the squared changes in our curve in all of these directions, we can rewrite our integral as the following expression:

$$\text{arc\_length}(\gamma) = \int_a^b \sqrt{\left(\frac{d\gamma_1}{dt}\right) + \left(\frac{d\gamma_2}{dt}\right) + \ldots + \left(\frac{d\gamma_n}{dt}\right)}\; dt.$$

As an example of this formula, consider the following example:

Example. Find the length of the helix $\gamma(t) = (\cos(t), \sin(t), t)$, where we let t range over $\in [0, 6\pi]$.

Solution. Because it's pretty, we sketch this curve here:



Ok! Now, with that done, we just simply apply our above definition:

$$\begin{aligned}
\int_\gamma ds &= \int_0^{6\pi} \sqrt{\left(\frac{d\gamma_1}{dt}\right)^2 + \left(\frac{d\gamma_2}{dt}\right)^2 + \left(\frac{d\gamma_3}{dt}\right)^2}\ dt \\
&= \int_0^{6\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2}\ dt \\
&= \int_0^{6\pi} \sqrt{\sin^2(t) + \cos^2(t) + 1}\ dt \\
&= \int_0^{6\pi} \sqrt{2}\ dt \\
&= 6\pi\sqrt{2}.
\end{aligned}$$

2 Div, Grad, and Curl

The other focus of this class was the twin definitions of **div** and **curl**, which we quickly review here:

1. **Div and curl: definitions.** Given a C^1 vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$, we can defind the **divergence** and **curl** of F as follows:

- **Divergence**. The **divergence** of F, often denoted either as $\text{div}(F)$ or $\nabla \cdot F$, is the following function $\mathbb{R}^3 \to \mathbb{R}$:

$$\text{div}(F) = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

- **Curl.** The **curl** of F, denoted $\text{curl}(F)$ or $\nabla \times F$, is the following map $\mathbb{R}^3 \to \mathbb{R}^3$:

$$\text{curl}(F) = \nabla \times F = \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \right).$$

  Often, the curl is written as the "determinant" of the following matrix:

$$\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

Given a function $F : \mathbb{R}^2 \to \mathbb{R}^2$, we can also find its curl by "extending" it to a function $F^\star : \mathbb{R}^3 \to \mathbb{R}^3$, where $F_1^\star(x, y, z) = F(x, y)$, $F_2^\star(x, y, z) = F(x, y)$, and $F_3^\star(x, y, z) = 0$. If someone asks you to find the curl of a function that's going from $\mathbb{R}^2 \to \mathbb{R}^2$, this is what they mean.

Also, divergence naturally generalizes to working on any function $\mathbb{R}^n \to \mathbb{R}^n$; just take the sum of $\frac{\partial F_i}{\partial x_i}$ over all of the variables the function depends on.

The **divergence** of a vector field at a point, physically speaking, can be interpreted as the net "flow" of our vector field through this point; i.e. kind-of the extent to which there is a net positive or negative amount of flow through our surface at this point. In other words, take a vector field on some space, and imagine drawing a tiny box around some point in our space. The **divergence** at that point is approximately the net flow through this tiny box. So, if you have a vector field that is modeling some sort of incompressible fluid (like, say, water), you would expect that the divergence at every point in your flow is 0: this is because an incompressible fluid locally has to have as much fluid going into each point as goes out (otherwise, you'd have compression or decompression.)

This isn't necessarily obvious from the definition of divergence at the moment; later in the class, however, when we prove the divergence theorem, this will make more sense.

The **curl** of a vector field also has a physical interpretation. Take a vector field on $\mathbb{R}^3$, and imagine that it is describing some sort of a fluid's motion. Take a point in $\mathbb{R}^3$, and fix some small ball at that point in space. Suppose that this ball has a ridged or rough outer surface; then, when our fluid flows past it, the fluid will catch on these rough outer parts and cause the ball to rotate.

The **curl** measures how this rotation works. Specifically, in any fixed vector field that corresponds to a flow, this ball will have some sort of fixed axis around which this rotation occurs. This axis is precisely the curl of the vector field at that point. The direction of this rotation is determined by the right-hand-rule, and the magnitude of this rotation is precisely half of the magnitude of the curl.

In particular, suppose we're calculating a curl of a vector field on $\mathbb{R}^2$, using the method we talked about above to extend the curl to $\mathbb{R}^2$. Visualize your vector field F as a depiction for how water is flowing on the surface of a pool, say. To visualize the curl, imagine placing a rubber ducky at some point in your pool. The axis of rotation of the duck will be

$$\begin{aligned}\operatorname{curl}(F) &= \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right) \\ &= \left(0 - 0, 0 - 0, \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right) \\ &= \left(0, 0, \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right).\end{aligned}$$

We can see this by using the fact that F, because it's a vector field on $\mathbb{R}^2$, neither depends on z (so the $\frac{\partial}{\partial z}$'s are all 0) nor has a z-component (so the ∂F_3's are all 0). So, in other words, our duck will rotate along the z-axis (which makes sense), with rotation and speed determined by $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$.

2. **Theorems**. We have a pair of rather useful theorems about the divergence and curl of functions, which we state here:

- For any C^2 function F, div(curl(F)) is always 0.
- For any C^2 function F, curl(grad(F)) is always 0.

These theorems are a pair of very useful tests that can often tell us that a given function F is not a conservative vector field (i.e. a gradient) or a curl of some other function. For example, if we examined the function $F(x, y, z) = (x, y, z)$, we can immediately tell that F is not the curl of some other function, because its divergence is $1 + 1 + 1 \neq 0$. Similarly, we can see that $V(x, y, z) = (xy, yz, xz)$ cannot be described as the gradient $\nabla(f)$ of some other function $f : \mathbb{R}^3 \to \mathbb{R}$, because its curl is $(-y, -z, -x)$, which is not equal to $(0, 0, 0)$.