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Recitation 3: Applications of the Derivative

Week 3 *Caltech 2013*

1 Higher-Order Derivatives and their Applications

Another thing we could want to do with the derivative, motivated by what we were able to do in $\mathbb{R}^1$, is the concept of **higher-order derivatives**. These are relatively easy to define for partial derivatives:

Definition. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, we can define its **second-order partial derivatives** as the following:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right).$$

In other words, the second-order partial derivatives are simply all of the functions you can get by taking two consecutive partial derivatives of your function f.

A useful theorem for calculating these partial derivatives is the following:

Theorem 1 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called C^2 at some point if all of its second-order partial derivatives are continuous at that point. If a function is C^2, then the order in which second-order partial derivatives are calculated* ***doesn't matter****: i.e.*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i},$$

for any i, j.

It bears noting that if the conditions of this theorem are **not** met, then the order for computing second-order partial derivatives may actually matter! One such example is the function

$$f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

At $(0,0)$, you can calculate that $\frac{\partial^2 f}{\partial x \partial y} = 1 \neq -1 = \frac{\partial^2 f}{\partial y \partial x}$: a result that occurs because the second-order partials of this function are not continuous.

However, the interesting aspects of higher-order partial derivatives are not really in their calculation; rather, the **applications** of higher-order partial derivatives are the things worth studying. In $\mathbb{R}$, for example, we could turn the second derivative of a function into a lot of information about that function: in particular, we could use this second derivative to determine

- whether a given critical point was a local minima or maxima,

- whether a function is concave up or down at a given point,
- and what the second-order Taylor approximation to that function was at a point.

Can we do the same for functions from $\mathbb{R}^n$ to $\mathbb{R}$? As it turns out, the answer is yes! The tool with which we do this is called the Hessian, which we define here:

Definition. The Hessian of a function $f : \mathbb{R}^n \to \mathbb{R}$ at some point $\mathbf{a}$ is the following matrix:

$$H(f)\big|_{\mathbf{a}} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\mathbf{a}) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{a}) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_n}(\mathbf{a}) \end{bmatrix}.$$

The main useful property of the Hessian is the following:

Theorem 2 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function with well-defined second-order partials at some point $\mathbf{a}$. Using the Hessian of f, construct the following function: $H(\mathbf{x}) = \frac{1}{2} \cdot (x_1, \dots x_n) \cdot H(f)\big|_{\mathbf{a}} \cdot (x_1, \dots x_n)^T$. Pick any two co ordinates x_i, x_j in $\mathbb{R}^n$: then*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}) = \frac{\partial^2 H}{\partial h_i \partial h_j}.$$

In other words, H's second-order partial derivative are precisely the second-order partial derivatives of f at $\mathbf{a}$! So H is basically a function designed to have the same second-order partials as f at $\mathbf{a}$.

One quick thing this theorem suggests is that we could use $H(f)\big|_{\mathbf{a}}$ to create a "second-order" approximation to f at $\mathbf{a}$, in a similar fashion to how we used the derivative to create a linear (i.e. first-order) approximation to f. We define this below:

Theorem 3 *If $f : \mathbb{R}^n \to \mathbb{R}$ is a function with continuous second-order partials, we define the **second-order Taylor approximation** to f at $\mathbf{a}$ as the function*

$$T_2(f)\big|_{\mathbf{a}}(\mathbf{a} + \mathbf{h}) = f(\mathbf{a}) + (\nabla f)(\mathbf{a}) \cdot \mathbf{h} + \frac{1}{2} \cdot (h_1, \dots h_n) \cdot H(f)\big|_{\mathbf{a}} \cdot (h_1, \dots h_n)^T.$$

You can think of $f(\mathbf{a})$ as the constant, or zero-th order part, $(\nabla f)(\mathbf{a}) \cdot \mathbf{h}$ as the linear part, and $H(f)\big|_{\mathbf{a}}(\mathbf{h})$ as the second-order part of this approximation.

To illustrate how this process actually creates a pretty decent approximation to f, we calculate an example:

Example. Calculate the second-order Taylor approximation to the function $f(x, y) = e^{xy}$ at the origin.

Answer. Calculating the second derivatives of f is pretty straightforward:

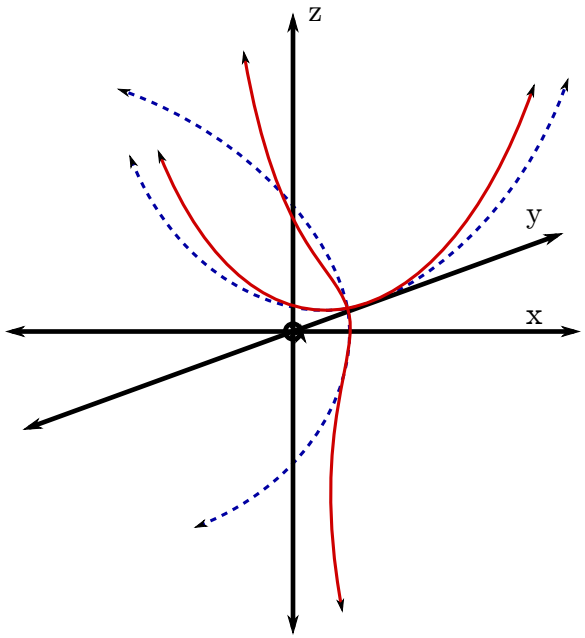
$$\frac{\partial f}{\partial x} = ye^{xy}, \frac{\partial f}{\partial y} = xe^{xy}$$

$$\frac{\partial^2 f}{\partial x^2} = y^2 e^{xy}, \frac{\partial^2 f}{\partial y^2} = x^2 e^{xy}, \frac{\partial^2 f}{\partial x \partial y} = xye^{xy} + e^{xy} = \frac{\partial^2 f}{\partial y \partial x}.$$

If we evaluate these partials at 0 and plug them into the definition above for $T_2(f)\big|_{(0,0)}$, we get

$$\begin{aligned} T_2(f)\big|_{(0,0)}((0,0)+(h_1,h_2)) &= f(0,0) + (\nabla f)(0,0)\cdot(h_1,h_2) + H(f)\big|_{(0,0)}(h_1,h_2) \\ &= 1 + (0,0)\cdot(h_1,h_2) + \frac{1}{2}(h_1,h_2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\ &= 1 + \frac{1}{2}(h_1,h_2)\begin{pmatrix} h_2 \\ h_1 \end{pmatrix} \\ &= 1 + \frac{1}{2}(2h_1h_2) \\ &= 1 + h_1h_2. \end{aligned}$$

So, at the origin, the second-order Taylor approximation for f is just $T_2(f)(x,y) = 1 + xy$. The following graph, with e^{xy} in solid red and T_2 in dashed blue, shows that it's actually a somewhat decent approximation at $(0,0)$:



As well, we can use the second derivatives to search for and find local minima and maxima! We define these terms here:

Definition. A point $\mathbf{a} \in \mathbb{R}^n$ is called a **local maxima** of a function $f : \mathbb{R}^n \to \mathbb{R}$ iff there is some small value r such that for any point $\mathbf{x}$ in $B_{\mathbf{a}}(r)$ not equal to $\mathbf{a}$, we have $f(\mathbf{x}) \leq f(\mathbf{a})$.

A similar definition holds for local minima.

So: how can we use the derivative to find such local maxima? Well, it's clear that (if our function is differentiable in a neighborhood around this point) that no matter how we move to leave this point, our function must not increase – in other words, for any direction $\mathbf{v} \in \mathbb{R}^n$, the directional derivative $f'(\mathbf{a}, \mathbf{v})$ must be ≤ 0. But this means that in fact all of

the directional derivatives must be **equal** to 0!, because if $f'(\mathbf{a}, \mathbf{v})$ was < 0, then $f'(\mathbf{a}, -\mathbf{v})$ would be > 0.

This motivates the following definitions, and basically proves the following theorem:

Definition. A point $\mathbf{a}$ is called a stationary point of some function $f : \mathbb{R}^n \to \mathbb{R}$ iff $\nabla(f)\Big|_{\mathbf{a}} = (0, \ldots, 0)$. A point $\mathbf{a}$ is called a **critical point** if it is a stationary point, or f is not differentiable in any neighborhood of $\mathbf{a}$.

Theorem 4 *A function $f : \mathbb{R}^n \to \mathbb{R}$ attains its local maxima and minima only at critical points.*

However, it bears noting that not every critical or stationary point is a local maxima or minima! A trivial example would be the function $f(x, y) = x^2 - y^2$: the origin is a stationary point, yet neither a local minima or maxima (as $f(0, \epsilon) < 0 < f(\epsilon, 0)$, and thus there are positive and negative values of f attained in any ball around the origin, where it is 0.)

How can we tell which stationary points do what? Well, in one-variable calculus, we used the idea of the "second derivative" to determine what was going on! In specific, we knew that if the second derivative of a function f at some point a was negative, then tiny increases in our variable at that point would cause the first derivative to decrease, and tiny decreases in our variable at that point would cause the negative of the first derivative to increase – i.e. cause the first derivative to decrease, and therefore make the function itself decrease! Therefore, the second derivative being negative at a stationary point implied that that point was a local maxima.

In higher dimensions, things are tricker – at a given point $\mathbf{a}$, we no longer have this idea of a "single" second derivative, but instead have many different second derivatives, like $\frac{\partial^2 f}{\partial x \partial y}(\mathbf{a})$ and $\frac{\partial^2 f}{\partial z^2}(\mathbf{a})$. Yet, we can still use the same ideas as before to figure out what's going on!

In particular, in one dimension, we said that we wanted tiny positive changes of our variables to make the first functions decrease. In other words, given any of the partials $\frac{\partial f}{\partial x_i}$, we want any positive changes in the direction of this partial to make our function decrease – i.e. we want the directional derivative of $\frac{\partial f}{\partial x_i}$ to be negative in any direction $\mathbf{v}$, where all of the coördinates of $\mathbf{v}$ are positive. (Positivity here stems from the same reason that in one dimension, we have that the first derivative is increasing for all of the points to the left of a maxima and decreasing for all of the points to the right of a maxima.)

So: this condition, if we write it out, is just asking that for every i and nonzero $\mathbf{v}$, that

$$\left(\frac{\partial^2 f}{\partial x_1 \partial x_i}(\mathbf{a}), \frac{\partial^2 f}{\partial x_2 \partial x_i}(\mathbf{a}), \ldots \frac{\partial^2 f}{\partial x_n \partial x_i}(\mathbf{a})\right) \cdot (v_1^2, v_2^2, \ldots v_n^2)$$

is negative. If you choose to write this out as a matrix, this actually becomes the claim that for any $\mathbf{v} \neq 0$, we have

$$\mathbf{v}^T \cdot \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\mathbf{a}) & \ldots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{a}) & \ldots & \frac{\partial^2 f}{\partial x_n \partial x_n}(\mathbf{a}) \end{bmatrix} \cdot \mathbf{v} < 0.$$

linear algebra, you may hopefully remember that any matrix satisfying this condition is called being **negative-definite**, and is equivalent to having all n of your eigenvalues existing and being negative. Similarly, if we were looking for a local minima, we would be asking that the above matrix product is always positive: i.e. that the matrix is **positive-definite**, which is equivalent to all of its eigenvalues being positive.

But we've seen this construction before! In particular, this matrix thing above is just the Hessian! Based on these observations, we make the following definitions and observations:

Definition. The Hessian $H(f)\big|_{\mathbf{a}}$ is positive-definite if and only if the matrix

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\mathbf{a}) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{a}) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_n}(\mathbf{a}) \end{bmatrix}$$

is positive-definite. (The same relation holds for being negative-definite.)

Recall from Math 1a that a matrix is positive-definite if and only if all of its eigenvalues are real and positive. Similarly, a matrix is negative-definite if and only if all of its eigenvalues are real and negative. If some of a matrix's eigenvalues are 0, some are negative and others are positive, or if there are less real eigenvalues than the rank of the matrix (i.e. some eigenvalues are complex,) then the matrix is neither positive-definite or negative-definite.

Note also that because the Hessian is symmetric whenever the mixed partials of our function are equal, and symmetric matrices have only real eigenvalues, you really should never get complex-valued eigenvalues.

Theorem 5 *A function $f : \mathbb{R}^n \to \mathbb{R}$ has a local maxima at a stationary point $\mathbf{a}$ if all of its second-order partials exist and are continuous in a neighborhood of $\mathbf{a}$, and the Hesssian of f is negative-definite at $\mathbf{a}$. Similarly, it has a local minima if the Hessian is positive-definite at $\mathbf{a}$. If the Hessian takes on both positive and negative values there, it's a **saddle point**: there are directions you can travel where your function increase, and others where it will decrease. Finally, if the Hessian is identically 0, you have no information as to what your function may be up to: you could be in any of the three above cases.*

A quick example, to illustrate how this gets used:

Example. For $f(x, y) = x^2 + y^2, g(x, y) = -x^2 - y^2$, and $h(x, y) = x^2 - y^2$, find local minima and maxima.

Solution. First, by taking partials, it is clear that the only point at which the gradient of these functions is $(0, 0)$ is the origin. There, we have that

$$H(f)\Big|_{(0,0)} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, H(g)\Big|_{(0,0)} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, H(h)\Big|_{(0,0)} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

and thus that f is positive-definite at (0,0), g is negative-definite at (0,0), and h is neither at (0,0) by examining the eigenvalues. Thus f has a local minima at $(0, 0)$, g has a local maxima at (0,0), and h has a saddle point at (0,0).

2 Lagrange Multipliers

2.1 Statement of the method.

In the section above, we talked about how to use derivatives to find and classify the **critical points** of functions $\mathbb{R}^n \to \mathbb{R}$. This allows us to find the global minima and maxima of functions over all of $\mathbb{R}^n$, if we want! Often, however, we won't just be looking to find the maximum of some function on all of $\mathbb{R}^n$: sometimes, we'll want to maximize a function *given a set of constraints*. For example, we might want to maximize the function $f(x, y, z) = x+y$ subject to the constraint that we're looking at points where $x^2 + y^2 = 1$. How can we do this?

Initially, you might be tempted to just try to use our earlier methods: i.e. look for places where Df is 0, and try to classify these extrema. The problem with this method, when we have a set of constraints, is that it usually **won't** find the maxima or minima on this constraint: because it's only looking for local maxima or minima over all of $\mathbb{R}^n$, it will ignore points that could be maxima or minima on our constrained surface! I.e. for the f, g we mentioned above, we know that $\nabla(f) = (1, 1)$, which is never 0; however, we can easily see by graphing that $f(x, y) = x + y$ should have a maximum value on the set $x^2 + y^2 = 1$, specifically at $x = y = \frac{1}{\sqrt{2}}$.

So: how can we find these maxima and minima in general? The answer is the method of Lagrange multipliers, which we outline here:

Proposition 6 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a function whose extremal values $\{\mathbf{x}\}$ we would like to find, given the constraints $g(\mathbf{x}) = c$, for some constraining function $g(\mathbf{x})$. Then, we have the following result: if $\mathbf{a}$ is an extremal value of f restricted to the set $S = \{\mathbf{x} : \forall i, g(\mathbf{x}) = c\}$, then either one of $\nabla(f)\big|_{\mathbf{a}}$ is 0, doesn't exist, or there is some constant λ such that*

$$\nabla(f)\big|_{\mathbf{a}} = \lambda \nabla(g)\big|_{\mathbf{a}}.$$

Why? In this case, it's worth talking a little bit about why this result happens to work, as understanding the proof of the above proposition is remarkably useful for using it. Consider, again, the example we discussed earlier, where we have

$$\begin{aligned} f(x, y) = x + y \qquad &\leftarrow \text{(the function we would like to maximize)}, \\ g(x, y) = x^2 + y^2 = 1 \qquad &\leftarrow \text{(our constraining function)}. \end{aligned}$$

Let

$$S = \{(x, y) : g(x, y) = 1\}.$$

In this notation, we are looking to maximize the function f restricted to the set S, which we denote $f\big|_S$. What do we know about $f\big|_S$? Well: if $\mathbf{a} \in S$ is a maximum, we would expect $\mathbf{a}$ to be a "critical point" of $f\big|_S$. The only issue is that we don't have any way to easily refer to just $f\big|_S$: we can talk about f in general, but if we don't restrict it to S we wouldn't expect $\mathbf{a}$ to still be a maximum.

One way around this is to think about **paths**. Specifically, pick any path γ such that γ's image is constained entirely within S, and $\gamma(0) = \mathbf{a}$. Then, if we look at $f \circ \gamma$, we know

that this is a function from $\mathbb{R} \to \mathbb{R}$; therefore, if $f\big|_S$ has a maximum at $\mathbf{a}$, $f \circ \gamma$ *also* must have a maximum, as it's just a path contained entirely in S that goes through this supposed maximum point $\mathbf{a}$.

In other words, we have

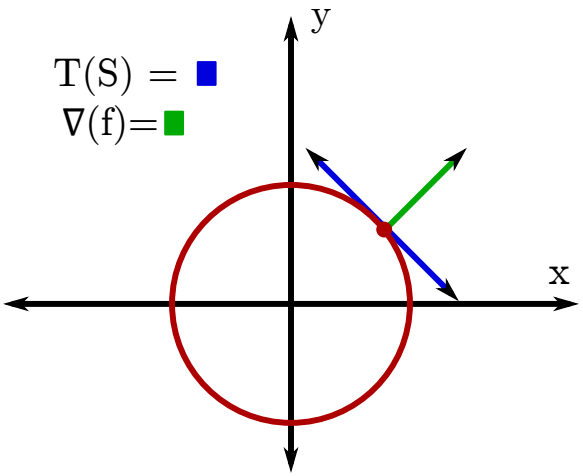
$$\begin{aligned} & \nabla(f \circ \gamma)\big|_{t=0} = 0 \\ \Rightarrow \quad & \nabla(f)\big|_{\mathbf{a}} \cdot \gamma'(0) = 0; \end{aligned}$$

i.e. $\nabla(f)\big|_{\mathbf{a}}$ is orthogonal to $\gamma'(0)$, for any path γ in S, going through 0. But these $\gamma'(0)$'s are just all of the possible tangent vectors to S at $\mathbf{a}$: so we have that $\nabla(f)\big|_{\mathbf{a}}$ is orthogonal to all of these tangent vectors!

Similarly, we know that for any such path γ, we have that $g \circ \gamma$ is constant, because g is constant on S. But this means that (because the derivative of any constant is 0)

$$\begin{aligned} & \nabla(g \circ \gamma)\big|_{t=0} = 0 \\ \Rightarrow \quad & \nabla(g)\big|_{\mathbf{a}} \cdot \gamma'(0) = 0. \end{aligned}$$

In other words, $\nabla(g)\big|_{\mathbf{a}}$ is *also* orthogonal to all of S's tangent vectors at $\mathbf{a}$!



But S is a space formed by placing one constraint on a function of n variables: in other words, it's a $n-1$-dimensional space! Therefore, at the point $\mathbf{a}$, the collection of tangent vectors to S at $\mathbf{a}$ is a $n-1$-dimensional space, contained in $\mathbb{R}^n$. But this means that the space of all vectors orthogonal to this $(n-1)$-dimensional space is a 1-dimensional space! In specific, we've just shown that both $\nabla(f)\big|_{\mathbf{a}}$ and $\nabla(g)\big|_{\mathbf{a}}$ are contained in the same 1-dimensional space: i.e. that one of them is a multiple of the other! In other words, we've shown that because they're both orthogonal to the entire tangent space to S at $\mathbf{a}$, there is some λ such that

$$\nabla(f)\big|_{\mathbf{a}} = \lambda \nabla(g)\big|_{\mathbf{a}}$$

(Or one of them is 0, or undefined.)

In the very specific case we're working with where

$$f(x,y) = x + y, g(x,y) = x^2 + y^2 = 1,$$

we have

$$\nabla(f(x,y)) = (1,1), \nabla(g(x,y)) = (2x, 2y)$$

and we're looking for points where either of these gradients are 0, or where there is some λ such that

$$\begin{aligned}
&\nabla(f(x,y)) = (1,1) = \lambda \cdot \nabla(g(x,y)) = (2\lambda x, \lambda y) \\
\Rightarrow &1 = 2\lambda x,\ 1 = 2\lambda y \\
\Rightarrow &\frac{1}{2\lambda} = x,\ \frac{1}{2\lambda} = y \\
\Rightarrow &x = y.
\end{aligned}$$

So: we have either $(0,0)$, as this forces $(2x, 2y) = (0,0)$, or points (x,y) where $x = y$. The first is impossible if we're looking at points where $g(x,y) = x^2 + y^2 = 1$; for the second, we would have $x^2 + x^2 = 1$, i.e $.x = y = \pm\frac{1}{\sqrt{2}}$.

We've therefore discovered the two possible extremal points of f: $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. In other words, we know that these points are the possible local maxima and minima of f. How do we tell whether these points are actually **global** minima and maxima? The answer is in the following brief definitions and theorem:

Definition. A set $S \subset \mathbb{R}^n$ is called **closed** if it contains all of its limit points: i.e. if $\{x_n\}_{n=1}^{\infty} \subset S,$, and $\lim_{n\to\infty} x_n = L$, then $L \in S$.

Definition. A set $S \subset \mathbb{R}^n$ is called **bounded** if there is some M such that $||\mathbf{x}|| < M$, for every $\mathbf{x} \in S$.

Lemma 7 *If $g : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and c is any constant, then the set $S = \{\mathbf{x} : g(\mathbf{x}) = c\}$ is closed.*

Theorem 8 *If f is a continuous function and we restrict f to a closed and bounded set S, then $f\big|_S$ will hit its global minima and maxima on S, and furthermore do this at critical points: i.e. places where $D(f\big|_S)$ is 0.*

Corollary 9 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is some function we want to maximize/minimize, $g : \mathbb{R}^n \to \mathbb{R}$ is some constraint function, S is the constrained set $\{\mathbf{x} : g(\mathbf{x}) = c\}$, and S is a bounded set. Then the absolute maxima and minima of g can all be found via the method of Lagrange multipliers: i.e. the maxima and minima of $f\big|_S$ will come up in the extremal points that the method of Lagrange multipliers finds.*

As a result of this theorem, we know that in our example earlier, one of the two points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ is the maximum of f, while the other must be its minimum. By plugging in both values to f, we can see that the first is the maximum, and the second is the minimum.

To illustrate the power and versatility of the method of Lagrange multipliers, and to help you get a better feel for how they work in practice, we work two examples using the tools we've just developed:

Example. Consider the **astroid**, a curve in $\mathbb{R}^2$ formed by the equation

$$x^{2/3} + y^{2/3} = 1.$$

What points on this curve are the closest to the origin?

Solution. We want to minimize the distance function

$$f(x, y) = \sqrt{x^2 + y^2}$$

given the constraint

$$g(x, y) = x^{2/3} + y^{2/3} = 1.$$

Using the method of Lagrange multipliers, we know that these minimal points will be those for which either $\nabla(f)$ or $\nabla(g)$ are undefined, or such that there is some λ such that

$$\nabla(f)\big|_{\mathbf{a}} = \lambda \nabla(g)\big|_{\mathbf{a}}.$$

So: calculating, we can see that

$$\nabla(f) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}\right),$$

which is defined whenever $(x, y) \neq (0, 0)$, and

$$\nabla(g) = \left(\frac{2}{3} \cdot x{-}1/3, \frac{2}{3} \cdot y{-}1/3, \right),$$

which is defined whenever $x \neq 0$ and $y \neq 0$.

When either x or $y = 0$, we know that (in order to satisfy $x^{2/3} + y^{2/3} = 1$) the other value has to be ± 1; so we immediately know that we should look at the four points $(\pm 1, 0), (0, \pm 1)$ when we're looking for our extremal points. Apart from these locations, we know that both of these gradients are well-defined and nonzero; so we're looking for values x, y, λ such that

$$\nabla(f) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}\right) = \lambda \nabla(g) = \left(\frac{2\lambda}{3} \cdot x{-}1/3, \frac{2\lambda}{3} \cdot y{-}1/3, \right).$$

By equating these two vectors, we're just trying to solve the two equations

$$\begin{aligned}
\frac{x}{\sqrt{x^2+y^2}} &= \frac{2\lambda}{3}\cdot x^{-1/3},\\
\frac{y}{\sqrt{x^2+y^2}} &= \frac{2\lambda}{3}\cdot y^{-1/3}\\
\Rightarrow \frac{1}{\sqrt{x^2+y^2}} &= \frac{2\lambda}{3}\cdot x^{-4/3},\\
\frac{1}{\sqrt{x^2+y^2}} &= \frac{2\lambda}{3}\cdot y^{-4/3}\\
\Rightarrow x^{-4/3} &= y^{-4/3}\\
\Rightarrow x &= \pm y.
\end{aligned}$$

The only points that satisfy $x = \pm y$ and also $x^{2/3}+y^{2/3}$ are the four points $\left(\pm\frac{1}{2\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right)$. Combining these with the $(\pm 1, 0), (0, \pm 1)$ points we discovered earlier, we have eight possible extremal points. Plugging these into $f(x)$ gives us $\frac{1}{2}$ for the points with $x = \pm y$ and 1 for the points with one of $x, y = 0$. Because the maximum and minimum values of f occur on these points, we know that the closest points to the origin are precisely the points $\left(\pm\frac{1}{2\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right)$.

Example. Consider the following rough model for the economics of pie-baking:

- Your ingredients for a pie are apples (a), butter (b), flour (f), and sugar (s).
- Suppose that apples cost \$2/unit, butter costs \$3/unit, flour costs \$1/unit, and sugar costs \$1/unit.
- Suppose that if you have a units of apples, b units of butter, f units of flour, and s units of sugar, you can make roughly $\sqrt[4]{abfs}$- many pies. (This is not an entirely implausible guess for a function that tells you how many pies you can make: in particular, you want a function that (1) is 0 whenever you don't have one of your ingredients, which taking the product of all of your ingredients does for you, and (2) grows linearly if you increase the quantity of each of your ingredients linearly. [i.e. if you have k units of each ingredient, this says you can make k pies, which seems accurate.] The formula also allows you to slightly skew the ingredient proportions of your pies: if apples are really expensive, you can have pies that have more dough to apples, whereas if sugar gets really expensive you can just increase the apple ratio.)
- Finally, suppose you start with 100 units of currency, and that you cannot have a negative amount of any of our ingredients (i.e. $a, b, f, s \geq 1$.)

What is the maximum number of pies you can make?

Solution. This is a bit different than our earlier problems. In particular, instead of just optimizing the function

$$F(a, b, f, s) = \sqrt[4]{abfs}$$

on one constraint, we are optimizing it over the **inequalities**

$$2a + 3b + f + s \leq 100, a \geq 0, b \geq 0, f \geq 0, s \geq 0.$$

How can we do this with Lagrange multipliers? Well: to do this, we just need to consider several cases. Specifically, suppose we have some maximum point (a, b, f, s). There are two possibilities:

1. This maximum point occurs on the interior of the set formed by our constraints $2a + 3b + f + s \leq 100, a \geq 0, b \geq 0, f \geq 0, s \geq 0$. Therefore, this point can be found by looking at $D(f)$, as it's a local maximum of f without any constraints!

2. Otherwise, this maximum point occurs on the boundary of the set formed by the constraints $2a + 3b + f + s \leq 100, a \geq 0, b \geq 0, f \geq 0, s \geq 0$. In other words, this maximum point occurs when we have either

   $$g(a, b, f, s) = 2a + 3b + f + s = 100,$$

   or when one of the four quantities a, b, f, s are 0. We can eliminate the cases where we have 0 of any of our quantity by just noticing that this trivially restricts us to making 0 pies, which is clearly not a maximum; this leaves us with just the above single constraint. But this is exactly the situation that Lagrange multipliers are set up to deal with! In particular, we can use Lagrange multipliers to maximize $\sqrt[4]{abfs}$ with respect to the constraint $g(a, b, f, s) = 2a + 3b + f + s = 100$.

Comparing all of the critical points we find in these ways will yield the overall maximum of $\sqrt[4]{abfs}$ on our entire set.

We perform these calculations here. First, because

$$\nabla(F) = \left(\frac{bfs}{4(abfs)^{3/4}}, \frac{afs}{4(abfs)^{3/4}}, \frac{abs}{4(abfs)^{3/4}}, \frac{abf}{4(abfs)^{3/4}} \right),$$

we can see that the gradient of F is only undefined or zero at places where some of its coördinates are zero or negative. Because our conditions require that $a, b, f, s \geq 0$, and in the case that any quantity is 0 we know that no pies are made, we know that this is impossible: therefore, we don't have to worry about f having any maxima on the interior of our set of constraints.

Now, we turn to the constraint $g(a, b, f, s) = 2a + 3b + f + s = 100$. Using Lagrange multipliers, we know that critical points will occur where

$$\nabla(F) = \left(\frac{bfs}{4(abfs)^{3/4}}, \frac{afs}{4(abfs)^{3/4}}, \frac{abs}{4(abfs)^{3/4}}, \frac{abf}{4(abfs)^{3/4}} \right) = \lambda \nabla(g) = (2\lambda, 3\lambda, \lambda, \lambda).$$

This occurs at points that satisfy the four equations

$$\frac{bfs}{4(abfs)^{3/4}} = 2\lambda, \frac{afs}{4(abfs)^{3/4}} = 3\lambda, \frac{abs}{4(abfs)^{3/4}} = \lambda, \frac{abf}{4(abfs)^{3/4}} = \lambda.$$

By combining the first two equations, we can see that

$$\begin{aligned}
&\frac{bfs}{4(abfs)^{3/4}} = 2\lambda, \frac{afs}{4(abfs)^{3/4}} = 3\lambda \\
\Rightarrow &b = 2\left(4\lambda\frac{1}{fs}(abfs)^{3/4}\right), a = 3\left(4\lambda\frac{1}{fs}(abfs)^{3/4}\right) \\
\Rightarrow &\frac{b}{2} = \frac{a}{3}.
\end{aligned}$$

Similarly, we can combine the middle two equations to get

$$\begin{aligned}
&\frac{afs}{4(abfs)^{3/4}} = 3\lambda, \frac{abs}{4(abfs)^{3/4}} = \lambda \\
\Rightarrow &f = 3\left(4\lambda\frac{1}{as}(abfs)^{3/4}\right), b = \left(4\lambda\frac{1}{as}(abfs)^{3/4}\right) \\
\Rightarrow &\frac{f}{3} = b,
\end{aligned}$$

and(similarly) the last two equations to get $f = s$. Combining these results, we can write our point as $(a, \frac{2a}{3}, 2a, 2a)$, and get that

$$\begin{aligned}
&2a + 3b + f + s = 2a + 2a + 2a + 2a = 100 \\
\Rightarrow &a = 12.5, (a, \frac{2a}{3}, 2a, 2a) = (12.5, 8.\overline{3}, 25, 25).
\end{aligned}$$

With these ingredient ratios, we can make

$$\sqrt[4]{12.5 \cdot 8.\overline{3} \cdot 25 \cdot 25)} \cong 16$$

pies; as this is the only critical point that gives a nonzero value, we know that it must be our maximum.