Math 1c — TA: Padraic Bartlett

Recitation 9: Green, Gauss and Stokes's Theorems

Week 9 — *Caltech 2012*

1 Random Question

A **Hadamard matrix** is a $n \times n$ matrix H, all of whose entries are either $+1$ or -1, such that any two rows of H are **orthogonal**: i.e. the dot products of any two distinct rows is 0, or equivalently any two distinct rows of H agree at precisely half of their entries. An example Hadamard matrix is presented below:

$$\begin{pmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{pmatrix}$$

- (Easy.) Find some Hadamard matrices.
- (Not hard.) Prove there are no $n \times n$ Hadamard matrices, where n is odd and ≥ 3.
- (A little harder.) Find infinitely many Hadamard matrices.
- (Open.) Find a 668×668 Hadamard matrix.
- (Also open.) Show that there is a $4n \times 4n$ Hadamard matrix, for every n.

2 Green's Theorem: Motivation, Statement and Examples

Today's lecture, like almost every lecture we've given this quarter, is about how we can extend a concept from one-dimensional calculus to higher dimensions. Throughout this course, we've already extended the concepts of limits, derivatives, several derivative techniques, integrals, and several integral techniques from $\mathbb{R}^1$ to $\mathbb{R}^n$; basically, whenever we've seen anything in single-variable calculus, we've been able to extend it to $\mathbb{R}^n$. Loosely speaking, there's really only one major theorem that we haven't extended yet: the **Fundamental Theorem of Calculus**, which stated that (for $f : \mathbb{R} \to \mathbb{R}$ a C^1 function)

$$\int_a^b \frac{d}{dx}(f(x))dx = f(b) - f(a).$$

In other words, knowing the behavior of the derivative over an interval is equivalent to knowing the function's original values at the endpoints of that interval. This, you may remember, was a remarkably powerful technique: in single-variable calculus, the FTC often allowed us to transform knowledge of the derivative (often a far simpler thing than the original function) over a region into the function's actual behavior on the boundary of this region, and vice-versa.

A natural question to ask, then, is whether we can extend this to higher dimensions. I.e. take a region $R \subset \mathbb{R}^2$, with boundary ∂R. Can we relate the behavior of a function on ∂R to the behavior of some sort of derivative on all of R?

As it turns out, we can! This is precisely Green's theorem; to state it formally, we first make the following two definitions.

Definition. A **simple closed curve** γ is a map $[a, b] \to \mathbb{R}^n$ such that

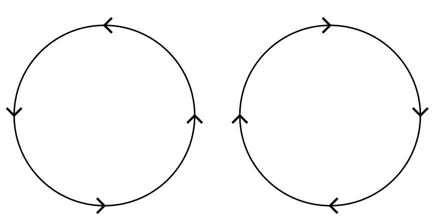
- $\gamma(a) = \gamma(b)$,
- γ has finite length, and
- γ does not intersect itself: i.e. for any two points $x \neq y \in [a, b]$, $\gamma(x) = \gamma(y)$ if and only if x and y are the two endpoints a, b.

Example. The following illustrates some closed curves that are simple, and some closed curves that are not simple:

(simple closed curves) (not simple closed curves)

Definition. Suppose that a simple closed curve γ is also the boundary of some region R. We say that a curve is **positively oriented** if travelling along our curve in the direction given by γ keeps R on the "left" of the curve. Similarly, a parametrization is **negatively oriented** if travelling along the curve keeps R on the "right."

Example. For example, the parametrization $\gamma_+(t) = (\cos(t), \sin(t))$ is a positively-oriented parametrization with respect to the unit disk. This is because moving along the unit disk using γ keeps the unit disk always on our left. Similarly, the parametrization $\gamma_-(t) = (\cos(t), -\sin(t))$ is negatively-oriented, because the unit disk is always on the right of our parametrization.



(positive) (negative)

Theorem 1 *(Green's Theorem.) Suppose that R is some region in $\mathbb{R}^2$ such that R's boundary is given by the curve C_1, and that γ is a positive parametrization of c_1. Suppose that P*

and Q are a pair of maps $\mathbb{R}^2 \to \mathbb{R}$ with continuous partial derivatives in an open neighborhood of R. Then, we have the following equality

$$\iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_\gamma Pdx + Qdy$$

3 Green's Theorem: Three Applications

Why do we care about Green's theorem? Well: from looking at its statement above, what does it do? It takes a pair of functions P, Q and sends an integral involving them to an integral involving their partials $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$; as well, it transforms a line integral over some curve C into a integral over some region R. This suggests that we might want to use Green's theorem in the following situations:

1. If we're integrating a pair of functions over some particularly awful curve, we might want to use Green's theorem to transform this integral into one over a region, in the hopes that the expression $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ might become zero or at the least a simpler expression.

2. Conversely, if we have a fairly awful region R, we might want to use Green's theorem to take us to a line integral, which can sometimes make our lives easier. One typical example of this is the use of Green's theorem to calculate the **area** of a region, which is the following equation:

$$\iint\limits_R 1\ dxdy = \frac{1}{2}\oint_C xdy - ydx.$$

   The left-hand side is (by definition) the area of the region R; the right-hand side is one possible pair of functions P, Q such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is 1.

We illustrate these two uses with two examples:

Example. For any two constants $a, b \in \mathbb{R}$, and $n \in \mathbb{N}$, find the integral

$$\oint_{C_n^+} a\cos(x)dx + b\sin(y)dy,$$

where C_n^+ is a counterclockwise-oriented n-gon with side length 1, center at (0,0), and one vertex on the x-axis.

Solution. So: this is (clearly) a case where our curve C_n^+ is far too awful to integrate along. Having no other option, we apply Green's theorem, which tells us that (if R is the region enclosed by our n-gon)

$$\begin{aligned}\oint_{C_n^+} a dx + b dy &= \iint\limits_R \left(\frac{\partial(b\cos(y))}{\partial x} - \frac{\partial(a\sin(x))}{\partial y} \right) dxdy \\ &= \iint\limits_R (0-0)\, dxdy \\ &= 0.\end{aligned}$$

Done!

Example. Find the area of the ellipse

$$R = \left\{ (x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}.$$

Solution. As mentioned before, the area of any region R can be given by the integral $\iint\limits_R 1\ dxdy$; so, if we choose $P(x,y) = -y/2, Q(x,y) = x/2$, we have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, and thus that

$$\iint\limits_R 1\ dxdy = \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \frac{1}{2} \oint_{C^+} xdy - ydx,$$

where C^+ is the boundary curve of our ellipse: i.e. $\gamma : [0, 2\pi] \to \mathbb{R}^2$, $\gamma(t) = (a\cos(t), b\sin(t))$.

Calculating, we have

$$\begin{aligned}\frac{1}{2} \oint_{C^+} xdy - ydx &= \frac{1}{2} \int_0^{2\pi} (-y, x)\Big|_{\gamma(t)} \cdot \gamma'(t) dt \\ &= \frac{1}{2} \int_0^{2\pi} (-b\sin(t), a\cos(t)) \cdot (-a\sin(t), b\cos(t)) dt \\ &= \frac{1}{2} \int_0^{2\pi} ab(\sin^2(t) + \cos^2(t)) dt \\ &= \frac{1}{2} \int_0^{2\pi} ab\ dt \\ &= ab\pi.\end{aligned}$$

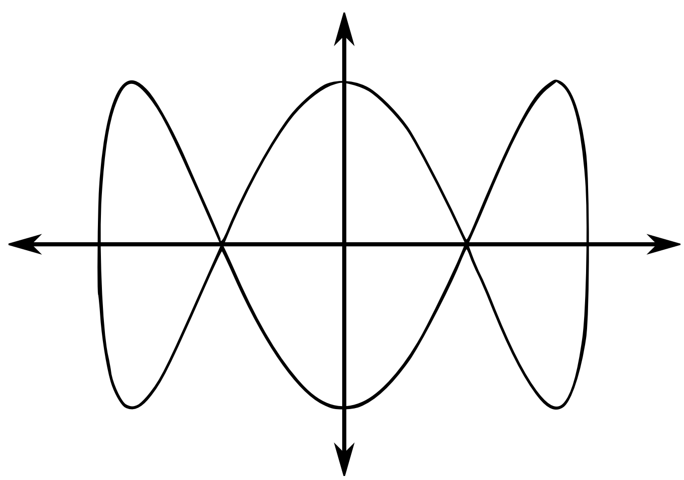
It bears noting that we had many possible choices of P, Q above! Specifically, we could have also chosen $Q = x, P = 0$; in this case, we would have had

$$\begin{aligned}
\iint\limits_R 1 \, dxdy &= \oint_{C^+} xdy \\
&= \int_0^{2\pi} (0, a\cos(t)) \cdot (-a\sin(t), b\cos(t))dt \\
&= \int_0^{2\pi} ab\cos^2(t)dt \\
&= \int_0^{2\pi} ab\frac{\cos(2t)+1}{2}dt \\
&= \left(ab\frac{\sin(2t)}{4} + \frac{abt}{2} \right) \Bigg|_0^{2\pi} \\
&= ab\pi.
\end{aligned}$$

This is the same answer! This is just an aside, to illustrate that you can have many different choices of P, Q available to you such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is equal to your desired expression.

The following example provides a slightly tricker area calculation, as well as a cautionary tale about making sure to always check your boundary conditions when you're applying a theorem:

Example. Find the area of the region R enclosed by the Lissajous curve $\gamma(t) = (\cos(t), \sin(3t))$, where t ranges from 0 to 2π.



Solution. When presented with a region R enclosed by a curve γ, it's really tempting to simply directly apply our Green's theorem for area result, which says that when γ is a simple closed curve oriented counterclockwise, we have

$$\text{area}(R) = \iint\limits_R 1dA = \int_\gamma \left(-\frac{y}{2}, \frac{x}{2} \right) d\gamma.$$

However, if we just directly apply this here, we'll get that

$$\int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right) d\gamma = \int_0^{2\pi} \left(-\frac{\sin(3t)}{2}, \frac{\cos(t)}{2}\right) \cdot (-\sin(t), 3\cos(3t))dt$$
$$= \frac{1}{2}\int_0^{2\pi} \sin(3t)\sin(t) + 3\cos(3t)(\cos(t)dt.$$

By applying your angle-addition formulas $\cos(3t) = \cos(t)\cos(2t) - \sin(t)\sin(2t), \sin(3t) = \sin(t)\cos(2t) + \sin(2t)\cos(t)$ (which, by the way, we don't expect you to have memorized), along with your double-angle formulas, we have that this is

$$\int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right) d\gamma = \frac{1}{2}\int_0^{2\pi} \sin(t)(\sin(t)\cos(2t) + \sin(2t)\cos(t)) + 3\cos(t)(\cos(t)\cos(2t) - \sin(t)\sin(2t))dt$$
$$= \frac{1}{2}\int_0^{2\pi} \sin^2(t)\cos(2t) + \sin(2t)\sin(t)\cos(t) + 3\cos^2(t)\cos(2t) - 3\sin(t)\cos(t)\sin(2t))dt$$
$$= \frac{1}{2}\int_0^{2\pi} \sin^2(t)\cos(2t) + 3\cos^2(t)\cos(2t) + \frac{\sin^2(2t)}{2} - \frac{3\sin^2(2t)}{2}dt$$
$$= \frac{1}{2}\int_0^{2\pi} \cos(2t) + 2\cos^2(t)\cos(2t) - \sin^2(2t)dt$$
$$= \frac{1}{2}\int_0^{2\pi} \cos(2t) + (1+\cos(2t))\cos(2t) - \frac{1-\cos(4t)}{2}dt$$
$$= \frac{1}{2}\int_0^{2\pi} 2\cos(2t) + \frac{1+\cos(4t)}{2} - \frac{1-\cos(4t)}{2}dt$$
$$= \frac{1}{2}\int_0^{2\pi} 2\cos(2t) + \cos(4t)dt$$
$$= 0.$$

Um. So, this is clearly false: our curve, by visual inspection, contains more area than 0. What went wrong? Well, our curve γ is **not** a simple closed curve: it has self-intersections!

So: to fix that, we can break up our curve γ into three parts:

- The part where γ's parameter t is restricted to the set $[-\pi/3, \pi/3]$. This is the far-right part of our curve; here, γ is counterclockwise-oriented, and we can thus find the area enclosed by γ by evaluating the integral

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3} 2\cos(2t) + \cos(4t)dt = \left.\frac{\sin(2t) + \sin(4t)/4}{2}\right|_{-\pi/3}^{\pi/3} = \frac{\sqrt{3}}{4}.$$

- The part where γ's parameter t is restricted to the set $[4\pi/3, 5\pi/3]$. This is the far-left part of our curve; here, γ is also counterclockwise-oriented, and we can thus find the area enclosed by γ by evaluating the integral

$$\frac{1}{2}\int_{4\pi/3}^{5\pi/3} 2\cos(2t) + \cos(4t)dt = \left.\frac{\sin(2t) + \sin(4t)/4}{2}\right|_{4\pi/3}^{5\pi/3} = \frac{\sqrt{3}}{4}.$$

- The part where γ's parameter t is restricted to the set$[\pi/3, 2\pi/3] \cup [4\pi/3, 5\pi/3]$. Here, γ is clockwise-oriented! Therefore, to find the area enclosed by gamma, we need to take the negative of this signed area, which is

$$\frac{1}{2}\int_{\pi/3}^{2\pi/3} 2\cos(2t) + \cos(4t)dt + \frac{1}{2}\int_{4\pi/3}^{5\pi/3} 2\cos(2t) + \cos(4t)dt = \ldots = \frac{\sqrt{3}}{2}.$$

  Notice that we've used a curve γ here that was piecewise defined: this is completely OK! The only thing you need to check is that the curve is a simple closed one and counterclockwise-oriented: once you've done that, it can be defined however you like.

Summing these three parts gives us that the area enclosed by our curve is $\sqrt{3}$.

3.1 Divergence / Stokes / Gauss's theorems.

There are several analogous results to Green's theorem for surfaces and volumes in $\mathbb{R}^2$ and $\mathbb{R}^3$; we list them here.

Theorem 2 *(Divergence theorem) Let R be a region in $\mathbb{R}^2$ with boundary given by some simple closed curve ∂R, and γ be a positive orientation of ∂R. Let $\mathbf{n}$ denote the outward point unit normal vector to the curve ∂R: i.e. set*

$$\mathbf{n} := \frac{(\gamma_2'(t), -\gamma_1'(t))}{||(\gamma_2'(t), -\gamma_1'(t))||}.$$

Then, if F is a smooth vector field on R, we have

$$\int_\gamma (F \cdot \mathbf{n}) d\gamma = \iint_R (\text{div}(F)) dA.$$

Theorem 3 *(Stokes' theorem.) Suppose that S is a bounded surface[1] with boundary given by the counterclockwise-oriented curve C, and $F : \mathbb{R}^3 \to \mathbb{R}^3$ is some continuously differentiable function. Then, we have the following equality:*

$$\iint_S ((\nabla \times F) \cdot n)\, dS = \int_C F \cdot ds,$$

where (as always) n denotes the unit normal vector at any point on S. (It bears noting that if we have a parametrization φ of our surface S, we can explicitly write this vector n as $\frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} / \left|\left| \frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} \right|\right|$.)

Theorem 4 *(Gauss's theorem.) Let W be a region in $\mathbb{R}^3$ with boundary given by some surface S. Let $\mathbf{n}$ be the outward-pointing unit normal vector to S, and let F be a smooth vector field defined on W. Then*

$$\iiint_W (\text{div}(F)) dV = \iint_{\partial W} (F \cdot \mathbf{n}) dS.$$

[1] A set S is called **bounded** if there is some n such that $||s|| < n$, for all $s \in S$

In practice, we use these theorems in pretty much all of the same cases that we use Green's theorem:

- Turning integrals of functions over really awful boundaries into integrals of curls or divergences over surfaces. Often, this process of taking a curl or divergence will make our function 0, or at the least quite trivial.
- If you're integrating something of the form $(\nabla \times f) \cdot n$ or $\text{div}(F) \cdot \mathbf{n}$ over a region, you can of course use the other direction of our proof to consider an integral over the boundary. In practice, this might not come up too often, as it's not always obvious when a given expression is a curl, or the dot product of a curl with a normal vector, or a divergence; so don't look for this unless you're really stuck, or the problem explictly gives you your function in one of these forms.

To illustrate how this goes, we work an example:

Example. If $F(x, y, z) = (-xy^2, x^2y, z)$ and S is the sphere cap $\{(x, y, z) : x^2 + y^2 + z^2 = 25, z \geq 4\}$, find the integral of $(\nabla \times F) \cdot n$ over S.

Solution. As you may have noticed from examples in the notes earlier, sphere caps are sometimes frustrating to work with. So, instead of integrating over this one, let's use Stokes' theorem to instead integrate along its boundary!

Specifically: the sphere cap above has boundary

$$\partial S = \{(x, y, z) : x^2 + y^2 + z^2 = 25, z = 4\} = \{(x, y, z) : x^2 + y^2 = 3^2, z = 4\},$$

which is traversed in the counterclockwise direction by the curve $\gamma(\theta) = (3\cos(\theta), 3\sin(\theta), 4)$. So, we can use Stokes' theorem to say that

$$\begin{aligned}
\iint_S (\nabla \times F) \cdot n dS &= \int_C F dc \\
&= \int_0^{2\pi} (-xy^2, x^2y, z)\Big|_{\gamma(\theta)} \cdot \gamma'(\theta) d\theta \\
&= \int_0^{2\pi} (-27\cos(\theta)\sin^2(\theta), 27\cos^2(\theta)\sin(\theta), 4) \cdot (-3\sin(\theta), 3\cos(\theta), 0) d\theta \\
&= \int_0^{2\pi} 81\cos(\theta)\sin^3(\theta) + 81\cos^3(\theta)\sin(\theta) d\theta \\
&= \int_0^{2\pi} 81\cos(\theta)\sin(\theta)(\sin^2(\theta) + \cos^2(\theta)) d\theta \\
&= \int_0^{2\pi} 81\cos(\theta)\sin(\theta) d\theta \\
&= \int_0^{2\pi} \frac{81\sin(2\theta)}{2} d\theta \\
&= 0.
\end{aligned}$$