Math 1c	TA: Padraic Bartlett

Recitation 6: Integration in $\mathbb{R}^n$

Week 6 — *Caltech 2012*

1 Random Question

2 Integration in $\mathbb{R}^n$

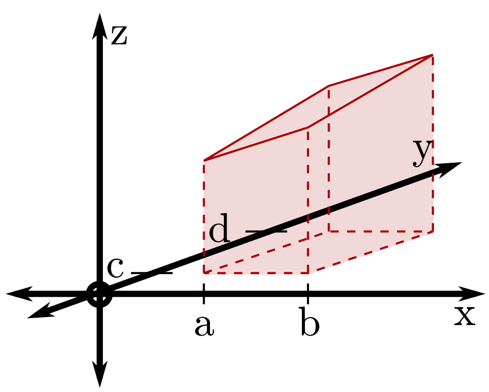
In the first five weeks of this course, we've introduced topics like limits, derivatives and optimization for functions on $\mathbb{R}^n$; furthermore, whenever we've done so, we've built all of our understanding and tools by looking at the 1-dimensional case, and extending our knowledge of functions on $\mathbb{R}^1$ to the study of $\mathbb{R}^n$. Today's lecture, on integration in $\mathbb{R}^n$, will be another class in this format!

In $\mathbb{R}^1$, we had two ways of looking at the definite integral of a function $f(x)$, $\int_a^b f(x)dx$. One was thinking of the integral as the **area under the curve** of $f(x)$ from a to b: in other words, the area of the region bounded by the lines $x = a, x = b, y = 0$ and the curve $f(x) = y$. Another, which we discussed a bit less, was the idea of the integral as the **average of** $f(x)$ from a to b, multiplied by the length of the interval $[a, b]$.

How can we "generalize" the integral to something we can calculate for functions $f : \mathbb{R}^n \to \mathbb{R}$? Well: first, we should generalize the concept of an interval $[a, b]$ to a box in $\mathbb{R}^n$: i.e. a region of the form $[a, b] \times [c, d] \times \ldots$ in $\mathbb{R}^n$. If we've done this, then the natural generalization of our "area" concept, at least for functions $\mathbb{R}^2 \to \mathbb{R}$, is the idea of **volume**: i.e. we can define the integral of $f(x, y)$ over some box $[a, b] \times [c, d]$,

$$\int_{[a,b]\times[c,d]} f(x, y)dA,$$

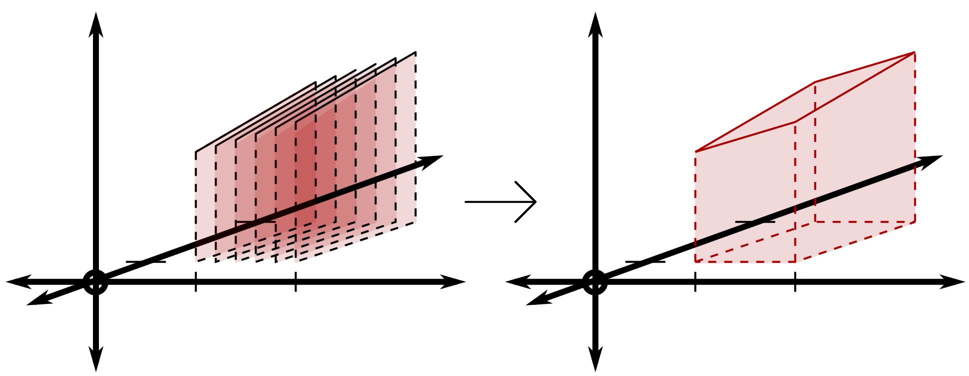
as simply the volume of the region bounded below by the plane $z = 0$, above by the surface $f(x, y) = z$, and with x, y coördinates constrained to the box $[a, b] \times [c, d]$. (The dA in the expression above is a reminder that we're integrating over a 2-dimensional region, and therefore that the "tiny bits" that we're using to integrate f are 2-dimensional, as opposed to one-dimensional like dx or dy.)



Similarly, if we'd rather extend the idea of "averages," we can think of the integral of a function in $f : \mathbb{R}^n \to \mathbb{R}$ over some box B as the average value of f over this entire box B, multiplied by the volume of the box B.

We now have a way to think about the integral in $\mathbb{R}^n$! The next natural thing to want to study, then, is how we'd ever actually go about *calculating* an integral: specifically, how we can use our past knowledge of integration in $\mathbb{R}^1$ to calculate integrals in $\mathbb{R}^n$. For the moment, let's think about functions $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$. If we're thinking of the integral in the above "volume" sense, then the following process makes sense as a possible way to calculate the integral in $\mathbb{R}^2$:

- Start with a function $f(x, y)$. We want to find the volume of the region between $z = f(x, y)$ and the plane $z = 0$, restricted to the box $[a, b] \times [c, d]$.
- To do this, pick any constant value $\lambda \in [a, b]$, and calculate the one-dimensional integral $\int_c^d f(\lambda, y)dy$. This is giving you the "area" of various cross-sections of the region we're studying, corresponding to the slices we get by setting x equal to said constant.
- Now, to combine all of these areas into a volume, simply integrate the function that spits out all of these areas, $\int_c^d f(x, y)dy$ (a function in one variable, x) over the interval $[c, d]$. The result is the average of these areas over the interval $[c, d]$, times the length of $[c, d]$: i.e. it's the **volume**!



Of course, we can also slice our surface by setting y constant first, and then integrating with respect to y: the picture will be the same. The upshot of all of this is that we can now calculate integrals in $\mathbb{R}^2$ using only integrals in $\mathbb{R}^1$: i.e. that

$$\int_{[a,b]\times[c,d]} f(x, y)dA = \int_a^b \left(\int_c^d f(x, y)dy \right) dx = \int_c^d \left(\int_a^b f(x, y)dx \right) dy.$$

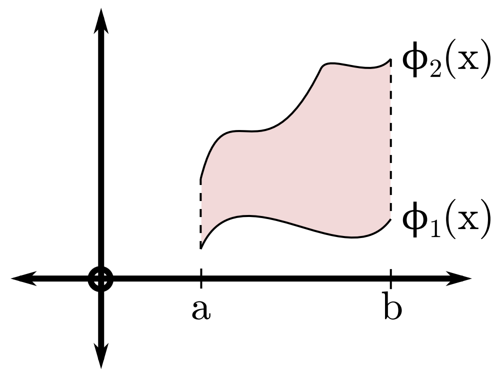
The above concept of the integral as an "average" will give you the exact same conclusion, as well. I generally prefer thinking of the integral as an average $\times$ the area of the thing we're integrating over, if only because it makes certain generalizations easier: i.e. using this concept, it's really easy to see that our above discussion for $\mathbb{R}^2$ generalizes completely to $\mathbb{R}^n$. For example, in $\mathbb{R}^3$, we can see that

$$\int_{[a,b]\times[c,d]\times[e,g]} f(x, y, z)dA = \int_a^b \left(\int_c^d \left(\int_e^g f(x, y, z)dz \right) dy \right) dx.$$

As well, if we want to integrate over regions that aren't boxes, this method generalizes beautifully! Let's think about functions $\mathbb{R}^2 \to \mathbb{R}$ for the moment. Suppose that we have a region D of the form

$$D = \{(x, y) : x \in [a, b], y \in [\phi_1(x), \phi_2(x)]\};$$

i.e. D looks like



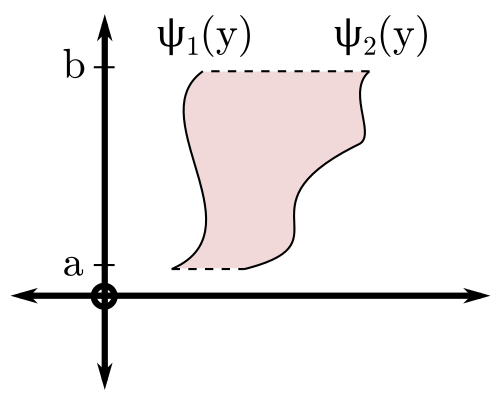
Then, if we want to calculate the integral of $f(x, y)$ over this region D, we'd want to perform the same slicing-into-areas trick as before: i.e. we would want to find the integrals of $f(\lambda, y)$ over $[\phi_1(\lambda), \phi_2(\lambda)]$ for every λ, and then we'd want to combine these areas into a volume by integrating with respect to x. In other words, we have

$$\int_D f(x, y)dA = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy \right) dx.$$

We call such regions x**-simple**, because they are regions that we can define by constraining x simply to an interval and then deriving curves that bound our choices of y. Similarly, a y**-simple** region D is one of the form

$$D = \{(x, y) : y \in [c, d], x \in [\psi_1(y), \psi_2(y)]\};$$

i.e. where D looks like



We can calculate integrals over a y-simple region just like we did for x-simple regions:

$$\int_D f(x,y)dA = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right) dy.$$

Sometimes, we can express a region D as both a x-simple and y-simple region! I.e. the upper-left quadrant of the unit circle can be expressed as both

$$\{(x,y) : x \in [0,1], y \in [0, \sqrt{1-x^2})]\}, \text{ and } \{(x,y) : y \in [0,1], x \in [0, \sqrt{1-y^2}]\}.$$

In regions where we can pull off this "two different descriptions" trick, we can often use these two different descriptions to study integrals that would be otherwise impossible. (In the examples this week, the third problem we study is precisely this case.)

To illustrate the methods we've talked about above, we perform a few example calculations:

Question 1 *Choose two random numbers from* $[0,1]$. *What is the average value of the smaller of the two? In other words, what is the integral (average value)*

$$\iint\limits_{[0,1]\times[0,1]} \min(x,y)dA \quad ?$$

Solution. Using our discussion above, start by expressing this integral as two nested one-dimensional integrals:

$$\iint\limits_{[0,1]\times[0,1]} \min(x,y)dA = \int_0^1 \left(\int_0^1 \min(x,y)dx \right) dy.$$

With this done, let's study the inner integral. Directly working with the function $\min(x,y)$ seems difficult. However, for a given fixed value of y, notice that we can split our integral into two parts (the integral from 0 to y and the integral from y to 1):

$$\int_0^1 \min(x,y)dx = \int_0^y \min(x,y)dx + \int_y^1 \min(x,y)dx$$

Then, if we notice that $\min(x,y)$ is just x whenever $x \leq y$ (i.e. x is in our first part) and is just y whenever $x \geq y$ (i.e. x is in our second part), we can replace the complicated

$\min(x, y)$'s with just x and y: i.e.

$$\begin{aligned}
\int_0^1 \min(x,y)dx &= \int_0^y \min(x,y)dx + \int_y^1 \min(x,y)dx \\
&= \int_0^y xdx + \int_y^1 ydx \\
&= \frac{x^2}{2}\Big|_0^y + xy\Big|_y^1 \\
&= \frac{y^2}{2} + y - y^2 \\
&= y - \frac{y^2}{2}.
\end{aligned}$$

Therefore, if we plug this into our nested integrals, we have that

$$\begin{aligned}
\iint\limits_{[0,1]\times[0,1]} \min(x,y)dA &= \int_0^1 \int_0^1 \min(x,y)dxdy \\
&= \int_0^1 \left(y - \frac{y^2}{2}\right) dy \\
&= \frac{y^2}{2} - \frac{y^3}{6}\Big|_0^1 \\
&= \frac{1}{3}.
\end{aligned}$$

So, if you take two random numbers in $[0, 1]$ and look at the smaller of the two, the average value you'd see is $1/3$.

Question 2 *Now, choose two random positive numbers so that their sum is between 0 and 1. What is the average value of the smaller of the two?*

Solution. Similarly to before, we want to integrate $\min(x, y)$ over some region: however, our region is now

$$D = \{(x, y) : 0 \leq x, 0 \leq y, x + y \leq 1\} = \{(x, y) : y \in [0, 1], x \in [0, 1 - y]\}.$$

Using this description of D, we can express our integral as the following two nested one-dimensional integrals:

$$\iint\limits_{[0,1]\times[0,1]} \min(x,y)dA = \int_0^1 \left(\int_0^{1-y} \min(x,y)dx\right) dy.$$

With this done, let's study the inner integral. It's now a bit more complicated! In specific, notice that if we have $y \geq \frac{1}{2}$, because $x \in [0, 1 - y]$, we will always have $x < y$: i.e.

$\min(x, y) = x$. So, for $y \geq \frac{1}{2}$, we have

$$\int_0^{1-y} \min(x, y)dx = \int_0^{1-y} xdx = \frac{x^2}{2}\Big|_0^{1-y} = \frac{(1-y)^2}{2} = \frac{y^2 - 2y + 1}{2}.$$

However, if $y \leq \frac{1}{2}$, it's possible that $x > y$ or $x < y$, and we are led to perform the same trick as before of splitting our integral into two pieces, one from 0 to y and the other from y to $1 - y$:

$$\begin{aligned}
\int_0^{1-y} \min(x, y)dx &= \int_0^{y} \min(x, y)dx + \int_y^{1-y} \min(x, y)dx \\
&= \int_0^{y} xdx + \int_y^{1-y} ydx \\
&= \frac{x^2}{2}\Big|_0^{y} + xy\Big|_y^{1-y} \\
&= \frac{y^2}{2} + (y - y^2) - y^2 \\
&= y - \frac{3y^2}{2}.
\end{aligned}$$

Therefore, if we plug these results into our nested integrals, we get that

$$\begin{aligned}
\iint\limits_{[0,1]\times[0,1]} \min(x, y)dA &= \int_0^1 \int_0^1 \min(x, y)dxdy \\
&= \int_0^{1/2} \left(y - \frac{3y^2}{2}\right) dy + = \int_{1/2}^{1} \left(\frac{y^2 - 2y + 1}{2}\right) dy \\
&= \left(\frac{y^2}{2} - \frac{y^3}{2}\right)\Big|_0^{1/2} + \left(\frac{y^3}{6} - \frac{y^2}{2} + \frac{y}{2}\right)\Big|_{1/2}^{1} \\
&= \left(\frac{1}{8} - \frac{1}{16}\right) - 0 + \left(\frac{1}{6} - \frac{1}{2} + \frac{1}{2}\right) - \left(\frac{1}{48} - \frac{1}{8} + \frac{1}{4}\right) \\
&= \frac{1}{12}
\end{aligned}$$

Therefore, if you take two random positive numbers such that their sum is ≤ 1 and you look at the smaller of the two, you'll get $\frac{1}{12}$ on average.

Our final problem illustrates when it can be useful to "switch" the order of integration, in order to deal with otherwise-difficult integrals:

Question 3 *Find*

$$\int_0^1 \int_y^1 \frac{\sin(x)}{x} dxdy.$$

Solution. First, notice that we cannot currently evaluate this integral as it's written: the function $\frac{\sin(x)}{x}$, the sinc function, is a well-studied object in mathematics that is notorious for (amongst other things) *not having* an indefinite integral that can be expressed in terms of elementary functions. In other words, there is no combination of $\sin, \cos, \tan$, polynomials, e^x and ln's or such things that will ever give you the antiderivative of $\frac{\sin(x)}{x}$; consequently, we really don't want to have to work with things that, um, require us to take the integral of this function!

Luckily for us, we don't have to! First, observe that the region we're integrating over (call it D) can be expressed in two different ways:

$$D = \{(x, y) : y \in [0, 1], x \in [y, 1]\} = \{(x, y) : x \in [0, 1], y \in [0, x]\}.$$

The easiest way to see this is to actually draw out what D is: according to both of the definitions above, D is precisely a right triangle with one side on the x-axis, another side being the line $y = 1$, and the third side given by the line $y = x$.

Using this, we can switch the order of integration in our integral above, to get

$$\int_0^1 \int_y^1 \frac{\sin(x)}{x} dxdy = \int_0^1 \int_0^x \frac{\sin(x)}{x} dydx.$$

This is now far easier to calculate! In particular, because $\frac{\sin(x)}{x}$ is a constant with respect to y, we get

$$\begin{aligned}
\int_0^1 \int_y^1 \frac{\sin(x)}{x} dxdy &= \int_0^1 \int_0^x \frac{\sin(x)}{x} dydx \\
&= \int_0^1 \left(y \frac{\sin(x)}{x} \right) \Bigg|_0^x dx \\
&= \int_0^1 (\sin(x) - 0) dx \\
&= -\cos(x) \Big|_0^1 \\
&= \cos(0) - \cos(1).
\end{aligned}$$

Success!