Math 1c	TA: Padraic Bartlett

Recitation 10: Final Review - Solutions to Sample Questions

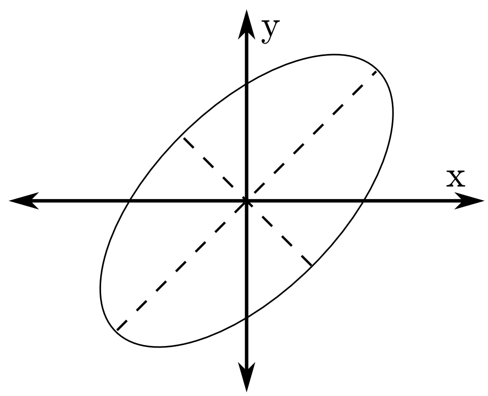
Week 10 *Caltech 2012*

Solutions to the practice questions for the final are posted below!

1. Consider the ellipse

$$\left(\frac{x-y}{2}\right)^2 + \left(\frac{x+y}{4}\right)^2 = 1,$$

depicted below.



What is the average value of the function $F(x, y) = x^2$ on the interior of this ellipse?

Solution. Denote the region given by the interior of our ellipse as R. We are looking for the average value of x^2 over R: in other words, we want to calculate the ratio

$$\frac{\iint\limits_R x^2 dxdy}{\iint\limits_R 1\ dxdy}.$$

To do this, we need a way to integrate over the region R: how can we do this? A direct approach seems difficult; our ellipse is not centered on the axes, and therefore we don't have any nice ways (like using polar) to parametrize it.

However, if we use a change of variables map, we can make this integral much easier! One particularly natural thing to try is to find a map T that sends the unit disk to our ellipse; if we can do this, then we can use our change-of-variables map to notice that

$$\iint\limits_R x^2 dxdy = \iint\limits_{\text{unit disk}} x^2\big|_{T(x,y)} \cdot \det(D(T))dxdy,$$

which should hopefully be easier (as the unit disk is a lot easier to work with.)

So: visually, suppose we're looking for a map that stretches and skews the unit disk into this ellipse. If we want this stretching and skewing to preserve the "circular" shape of our unit disk, we'd want our map to be **linear**: i.e. it should be of the form $T(x, y) = (ax + by, cx + dy)$. As well, we want it to map the unit circle onto this **specific** ellipse: i.e. we want to send the endpoint $(1, 0)$ of the x-axis of the unit circle onto the semimajor axis of the ellipse (i.e. the point $(2, 2)$ at the end of our dashed line in the upper-right quadrant), as well as sending the endpoint $(0, 1)$ of the y-axis of our unit circle to the semiminor axis $(-1, 1)$ (i.e. the point at the end of the dashed line in the upper-right quadrant.)

In other words, we want a map of the form

$$T(x, y) = (ax + by, cx + dy), T(1, 0) = (2, 2), T(0, 1) = (-1, 1);$$

i.e. $T(x, y) = (2x - y, 2x + y)$. Quickly double-checking, we can see that if we take any point $(2x - y, 2x + y)$ such that $x^2 + y^2 = 1$, we have that

$$\left(\frac{(2x - y) - (2x + y)}{2}\right)^2 + \left(\frac{(2x - y) + (2x + y)}{4}\right)^2 = y^2 + x^2 = 1;$$

so this map does indeed send points on the unit circle to the unit disk, as claimed.

So: now that we know how to change our coördinates, we can use this to integrate! Specifically, we have that

$$\begin{aligned}
\iint\limits_{R} x^2 dxdy &= \iint\limits_{\text{unit disk}} x^2\Big|_{T(x,y)} \cdot \det(D(T)) dxdy \\
&= \iint\limits_{\text{unit disk}} (2x - y)^2 \cdot \det\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} dxdy \\
&= 4 \iint\limits_{\text{unit disk}} (2x - y)^2 dxdy \\
&= 4 \int_0^1 \int_0^{2\pi} (2r\cos(\theta) - r\sin(\theta))^2 r dr d\theta \\
&= 4 \int_0^1 \int_0^{2\pi} r^3(4\cos^2(\theta) - 4\cos(\theta)\sin(\theta) + \sin^2(\theta)) dr d\theta \\
&= 4 \int_0^1 \int_0^{2\pi} r^3(3\cos^2(\theta) - 2\sin(2\theta) + 1) dr d\theta \\
&= 4 \int_0^1 \int_0^{2\pi} r^3 \left(3\frac{1 + \cos(2\theta)}{2} - 2\sin(2\theta) + 1\right) dr d\theta \\
&= 4 \int_0^1 \int_0^{2\pi} r^3 \left(\frac{5}{2} + \frac{3\cos(2\theta)}{2} - 2\sin(2\theta)\right) dr d\theta
\end{aligned}$$

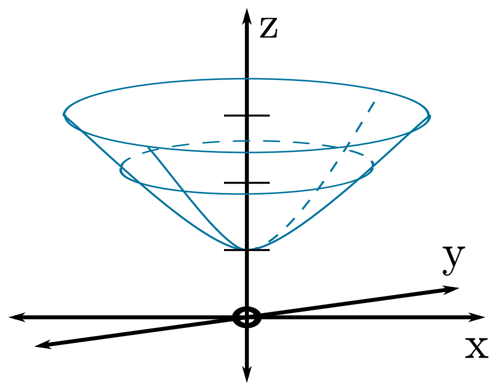
(Notice that we used a polar coördinate shift in this calculation, as well as some applications of the double-angle formulas for sin and cos.) Because the integral of $\sin(2x)$ or $\cos(2x)$ from 0 to 2π is zero (this is visually obvious, from inspecting the graph of sin or cos), we can ignore these terms; therefore, our integral is just the integral $\int_0^1 \int_0^{2\pi} 10r^3 dr d\theta$, which is just 5π.

Similarly, you can calculate the integral $\iint_R 1 dxdy$, or you can just remember the formula for the area of a ellipse, πab where a, b are the lengths of the major and minor axes, to see that this integral is $\pi \cdot ||(-1,1)|| \cdot ||(2,2)|| = \pi \cdot \sqrt{2} \cdot 2\sqrt{2} = 4\pi$. Therefore, our ratio is $\frac{5}{4}$.

2. Suppose that S is the upper sheet of the hyperboloid with two sheets

$$z^2 - x^2 - y^2 = 1, z \in [1, 3],$$

depicted below:



Let $F(x, y, z) = (-x, -y, -z)$ denote a vector field modelling a snowfall: i.e. at any point $(-x, -y, -z)$ in $\mathbb{R}^3$, F indicates the magnitude of snow flowing through $(-x, -y, -z)$, along with the direction it flows in.

Assume that F is denoting inches of snow accumulated per hour. How much snow accumulates on our surface S in a hour: i.e. what is the total flow $\iint_S F \cdot dS$?

Solution. First, notice that because z is always greater than 0, we can solve for z and express our surface as the collection of all points (x, y, z) such that

$$z = \sqrt{1 + x^2 + y^2}, z \in [1, 3].$$

In other words, we can parametrize our surface via the map

$$U(x, y) = (x, y, \sqrt{1 + x^2 + y^2}),$$

where we let x, y range over all values that keep the z-coördinate in $[1, 3]$. A perhaps easier way to view this map is to use cylindrical coördinates (which the $x^2 + y^2$ term suggests to us as a good idea): i.e. to instead use the parametrization

$$T(r, \theta) = (r\cos(\theta), r\sin(\theta), \sqrt{1 + r^2}),$$

where r ranges from 0 to $2\sqrt{2}$ (as this lets z range from 1 to 3) and θ ranges from 0 to 2π.

If we do this, then we're just asking for the integral

$$\iint_S (x,y,z)\cdot dS = \int_0^{2\pi}\int_0^{2\sqrt{2}} (-r\cos(\theta), -r\sin(\theta), -\sqrt{1+r^2})\cdot(T_\theta \times T_r)drd\theta,$$

where we're calculating our flow through the surface with the orientation given by $(T_\theta \times T_r)$.

So: specifically, we have

$$\begin{aligned}(T_\theta \times T_r) &= (-r\sin(\theta), r\cos(\theta), 0)\times\left(\cos(\theta), \sin(\theta), \frac{r}{\sqrt{1+r^2}}\right)\\ &= \left(\frac{r^2\cos(\theta)}{\sqrt{1+r^2}}, \frac{r^2\sin(\theta)}{\sqrt{1+r^2}}, -r\right),\end{aligned}$$

which by plugging in $r=1, \theta=0$ we can see is the normal that points "outwards" from our hyperboloid cup / down along the z-axis.

If we plug in this normal into our integral, we get

$$\begin{aligned}\iint_S (x,y,z)\cdot dS &= \int_0^{2\pi}\int_0^{2\sqrt{2}} (-r\cos(\theta), -r\sin(\theta), -\sqrt{1+r^2})\cdot\left(\frac{r^2\cos(\theta)}{\sqrt{1+r^2}}, \frac{r^2\sin(\theta)}{\sqrt{1+r^2}}, -r\right)drd\theta\\ &= \int_0^{2\pi}\int_0^{2\sqrt{2}}\left(\frac{-r^3\cos(\theta)-r^3\sin(\theta)}{\sqrt{1+r^2}} + r\sqrt{1+r^2}\right)drd\theta\\ &= \int_0^{2\pi}\int_0^{2\sqrt{2}}\left(\frac{-r^3}{\sqrt{1+r^2}} + r\sqrt{1+r^2}\right)dr\\ &=2\pi\int_0^{2\sqrt{2}}\left(\frac{-r^3}{\sqrt{1+r^2}} + r\sqrt{1+r^2}\right)drd\theta\end{aligned}$$

If we make the substitution $u=1+r^2, du=2rdr, u-1=r^2$, we get that our integral is just

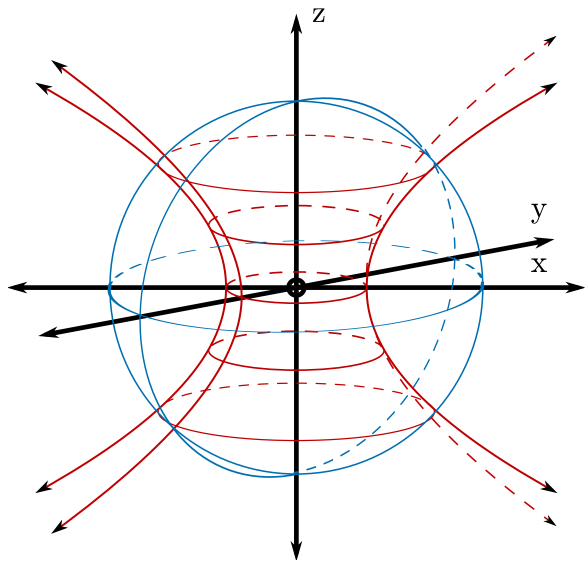
$$\begin{aligned}\pi\int_1^9\left(\frac{-(u-1)}{\sqrt{u}}+\sqrt{u}\right)du &= \pi\int_1^9\left(-\sqrt{u}+\frac{1}{\sqrt{u}}+\sqrt{u}\right)du\\ &= \pi\int_1^9 \frac{1}{\sqrt{u}}du\\ &= 2\pi\sqrt{u}\Big|_1^9\\ &= 4\pi.\end{aligned}$$

So we have 4π units of snow per hour that will flow through our surface with this outward/downward pointing normal: in other words, 4π units of snow per hour will accumulate in the top of our hyperboloid cup.

3. Let S be the surface given by taking the portion of the hyperboloid of one sheet

$$H_1 = \{(x, y, z) : x^2 + y^2 - z^2 = 1\}$$

contained by the sphere of radius 4, as depicted below:



Set up (but don't calculate) an integral for the surface area of S.

Solution. So: our surface is the collection of all points (x, y, z) such that

$$x^2 + y^2 = 1 + z^2, x^2 + y^2 \leq 16 - z^2.$$

The intersection of this sphere and hyperboloid occurs when these two constraints intersect: i.e. when we satisfy both $x^2 + y^2 = 1 + z^2$ and $x^2 + y^2 = 16 - z^2$. If we solve for z, we can see that this forces $1 + z^2 = 16 - z^2$: i.e. $z = \pm\sqrt{15/2}$. Therefore, by solving for z, we can parametrize the top half of our surface S via the map

$$T(x, y) = (x, y, \sqrt{x^2 + y^2 - 1}),$$

where we restrict the z-coördinate to lie between 0 and $\sqrt{15/2}$. More simply, if we use cylindrical coördinates, we have that this is just

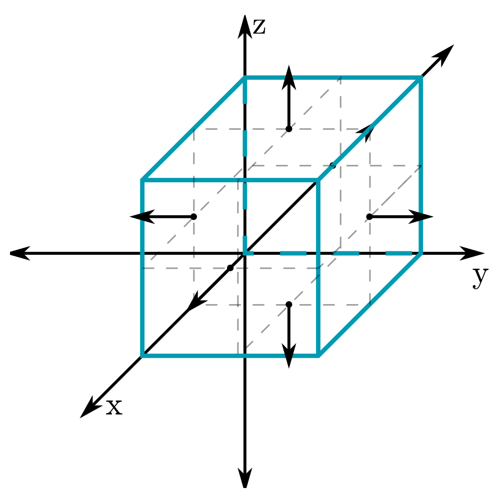
$$T(r, \theta) = (r\cos(\theta), r\sin(\theta), \sqrt{r^2 - 1}),$$

where r ranges from 1 to $\sqrt{17/2}$ and θ ranges from 0 to 2π. Therefore, our integral is just

$$\begin{aligned}
\iint_S 1 dS &= \int_1^4 \int_0^{2\pi} 1 \cdot ||T_r \times T_\theta|| \, d\theta dr \\
&= \int_1^{\sqrt{17/2}} \int_0^{2\pi} \left|\left| \left(\cos(\theta), \sin(\theta), \frac{r}{\sqrt{r^2-1}}\right) \times (-r\sin(\theta), r\cos(\theta), 0) \right|\right| d\theta dr \\
&= \int_1^{\sqrt{17/2}} \int_0^{2\pi} \left|\left| \left(-\frac{r^2\cos(\theta)}{\sqrt{r^2-1}}, -\frac{r^2\sin(\theta)}{\sqrt{r^2-1}}, r\right) \right|\right| d\theta dr \\
&= \int_1^{\sqrt{17/2}} \int_0^{2\pi} \sqrt{\frac{r^4\cos^2(\theta)}{r^2-1} + \frac{r^4\sin^2(\theta)}{r^2-1} + r^2 d\theta dr} \\
&= \int_1^{\sqrt{17/2}} \int_0^{2\pi} \sqrt{\frac{r^4}{r^2-1} + r^2 d\theta dr} \\
&= 2\pi \int_1^{\sqrt{17/2}} r\sqrt{\frac{2r^2-1}{r^2-1}} dr.
\end{aligned}$$

Therefore, our area is just two times that quantity (because as noted above, we restricted ourselves to looking at the top half in order to get an easier parametrization.)

4. Directly calculate the integral of $F(x, y, z) = (3x^2y, -3xy^2, z)$ over the surface of the unit cube, using the orientation depicted below. Then, use the divergence theorem to calculate this in a much faster manner.



Solution. If we want to do this directly, break the unit cube into its six sides

$$\begin{gathered}
[0,1] \times [0,1] \times \{0\}, [0,1] \times [0,1] \times \{1\}, \\
[0,1] \times \{0\} \times [0,1], [0,1] \times \{1\} \times [0,1] \\
\{0\} \times [0,1] \times [0,1], \{1\} \times [0,1] \times [0,1],
\end{gathered}$$

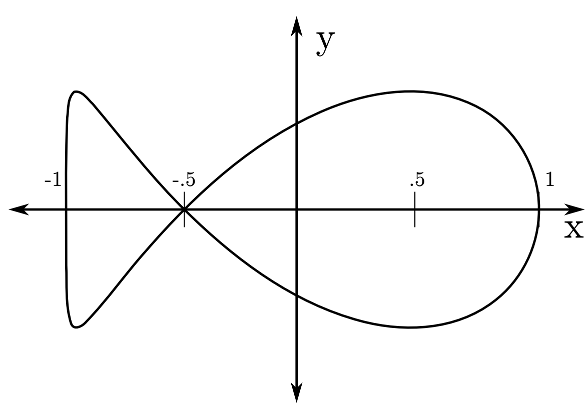
notice that the normals to these sides are precisely the normals $(0,0,\pm 1),(0,\pm 1,0),(\pm 1,0,0)$ as depicted in the above diagram, and calculate

$$
\begin{aligned}
&\iint\limits_{\text{surface of cube}} F \cdot dS \\
&= \int_0^1 \int_0^1 F\big|_{(x,y,0)} \cdot (0,0,-1)dxdy + \int_0^1 \int_0^1 F\big|_{(x,y,1)} \cdot (0,0,1)dxdy \\
&\quad + \int_0^1 \int_0^1 F\big|_{(x,0,z)} \cdot (0,-1,0)dxdz + \int_0^1 \int_0^1 F\big|_{(x,1,z)} \cdot (0,1,0)dxdz \\
&\quad + \int_0^1 \int_0^1 F\big|_{(0,y,z)} \cdot (-1,-0,0)dydz + \int_0^1 \int_0^1 F\big|_{(1,y,z)} \cdot (1,0,0)dydz \\
&= \int_0^1 \int_0^1 0dxdy + \int_0^1 \int_0^1 1dxdy + \int_0^1 \int_0^1 0dxdz + \int_0^1 \int_0^1 -3x\ dxdz \\
&\quad + \int_0^1 \int_0^1 0dydz + \int_0^1 \int_0^1 3ydydz \\
&=1.
\end{aligned}
$$

Alternately, if you use the divergence theorem, we can calculate this in a much faster way:

$$
\begin{aligned}
\iint\limits_{\text{surface of cube}} F \cdot dS &= \iiint\limits_{\text{cube}} (\mathrm{div} F)dV \\
&= \int_0^1 \int_0^1 \int_0^1 (6xy - 6xy + 1)dxdydz \\
&= \int_0^1 \int_0^1 \int_0^1 1dxdydz = 1.
\end{aligned}
$$

5. Let $c(t) = \left(\cos(t) - \frac{\sin^2(t)}{2}, \cos(t)\sin(t)\right)$ denote the "fish curve" drawn below:



Find the area contained within this curve.

Solution.

This looks like a textbook example of when to use the Green's theorem formula for the area contained in a curve. Specifically, Green's theorem, as applied to finding the area contained within a curve, says that if a region R is bounded by some simple closed curve $c(t)$ that is oriented positively (i.e. so that R is on the left as we travel along $c(t)$), then

$$\text{area}(R) = \iint_R 1dxdy \overbrace{=}^{\text{Green's theorem}} = \frac{1}{2}\int_{c(t)} (-y, x)\, dc.$$

If we just plug in our curve, we get that this integral is

$$\begin{aligned}
&\frac{1}{2}\int_{c(t)} (-y, x)\, dc \\
=&\frac{1}{2}\int_0^{2\pi} \left(-\cos(t)\sin(t), \cos(t) - \frac{\sin^2(t)}{2}\right)\cdot\left(-\sin(t) - \sin(t)\cos(t), \cos^2(t) - \sin^2(t)\right) dt \\
=&\frac{1}{2}\int_0^{2\pi} \left(\cos(t)\sin^2(t) + \cos^2(t)\sin^2(t) + \cos^3(t) - \cos(t)\sin^2(t) - \frac{cos^2(t)\sin^2(t)}{2} + \frac{\sin^4(t)}{2}\right) dt \\
=&\frac{1}{2}\int_0^{2\pi} \left(\frac{cos^2(t)\sin^2(t)}{2} + \cos^3(t) + \frac{\sin^4(t)}{2}\right) dt \\
=&\frac{1}{2}\int_0^{2\pi} \left(\frac{\sin^2(2t)}{8} + \cos(t)(1 - sin^2(t)) + \frac{(1-\cos(2t))^2}{8}\right) dt \\
=&\frac{1}{2}\int_0^{2\pi} \left(\frac{1-\cos(4t)}{16} + \cos(t)(1 - sin^2(t)) + \frac{1 - 2\cos(2t) + \cos^2(2t)}{8}\right) dt \\
=&\frac{1}{2}\int_0^{2\pi} \left(\frac{1-\cos(4t)}{16} + \cos(t)(1 - sin^2(t)) + \frac{1 - 2\cos(2t)}{8} + \frac{1+\cos(4t)}{16}\right) dt \\
=&\frac{1}{2}\int_0^{2\pi} \left(\frac{1}{4} + \cos(t)(1 - \sin^2(t)) - \frac{\cos(2t)}{4}\right) dt \\
=&\frac{1}{2}\left(\frac{t}{4} + \sin(t) - \frac{\sin^3(t)}{3} - \frac{\sin(2t)}{8}\right)\Big|_0^{2\pi} \\
=&\pi/4.
\end{aligned}$$

But is this plausible? Well: looking at our fish curve, it seems to contain at least (in the head-part) the area of an ellipse from $-.5$ to 1 with y-height from $-.5$ to $.5$, which is much greater than the area of a circle with radius .5, which is $\pi/4$. So: something has gone wrong!

What, specifically? Well: to apply Green's theorem, we needed a **simple closed curve** that was positively oriented. Did we have that here? No! In fact, our curve c has a self-intersection: $c(2\pi/3) = c(4\pi/3)$, and in fact the tail part of our curve is oriented negatively (i.e. if we travel around our curve from $2\pi/3$ to $4\pi/3$, our region is on the right-hand side. In fact, we've calculated the area of the head minus the area in the tail!

To calculate what we want, we want to take the integral above evaluated from $-\pi/2$ to $\pi/2$ (the head) and then add the integral from $3\pi/2$ to $\pi/2$ (travelling backwards here makes it so that we get the right orientation on the tail. Specifically, we have

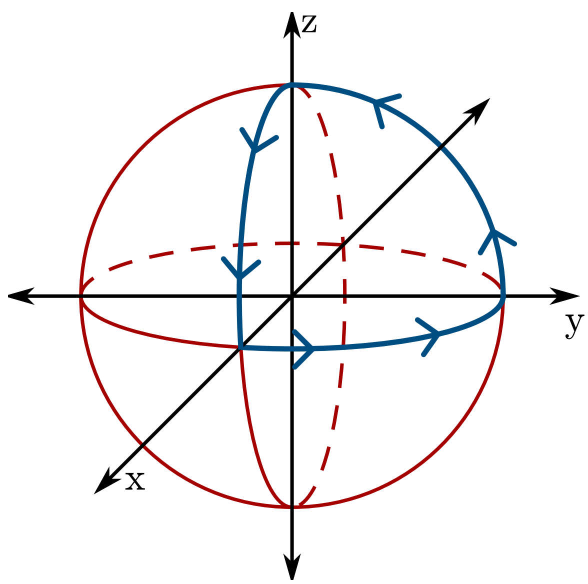
$$\begin{aligned}\frac{1}{2}\left(\frac{t}{4}+\sin(t)-\frac{\sin^3(t)}{3}-\frac{\sin(2t)}{8}\right)\Big|_{-\pi/2}^{\pi/2} =&\frac{1}{2}\left(\frac{\pi}{8}-\frac{-\pi}{8}+1-(-1)+(-\frac{1}{3})-\frac{1}{3}+0-0\right)\\ =&\frac{\pi}{8}+\frac{4}{3},\end{aligned}$$

while

$$\begin{aligned}\frac{1}{2}\left(\frac{t}{4}+\sin(t)-\frac{\sin^3(t)}{3}-\frac{\sin(2t)}{8}\right)\Big|_{3\pi/2}^{\pi/2} =&\frac{1}{2}\left(-\frac{\pi}{8}-\frac{\pi}{8}+1-(-1)+(\frac{-1}{3})-\frac{1}{3}+0-0\right)\\ =&-\frac{\pi}{8}+\frac{4}{3};\end{aligned}$$

therefore, our total area is $\frac{\pi}{8}+\frac{4}{3}+-\frac{\pi}{8}+\frac{4}{3}=\frac{8}{3}$.

6. Let $S=\{(x,y,z): x^2+y^2+z^2=1, x,y,z\geq 0\}$ and $C^+=\partial S$ be the boundary of S traversed in the counterclockwise direction from high above the z-axis, as depicted below:



Let $F(x,y,z)=(x^4,y^4,z^4)$ be a vector field. Calculate $\int_{C^+} F\cdot dc$ directly, then use Stokes's theorem to calculate it with much less effort.

Solution. To calculate this directly, first parametrize C as the three curves $\gamma_1,\gamma_2,\gamma_3$, where

$$\begin{aligned}\gamma_1(t)&=(\cos(t),\sin(t),0),\\ \gamma_2(t)&=(0,\cos(t),\sin(t)),\\ \gamma_3(t)&=(\sin(t),0,\cos(t)),\end{aligned}$$

and t ranges from 0 to $\pi/2$ for each curve.

Then, we'd have that

$$\begin{aligned}
\int_C F dC &= \sum_{i=1}^{3} \int_0^{\pi/2} (F \circ \gamma_i(t)) \cdot (\gamma'(t)) dt \\
&= \sum_{i=1}^{3} \int_0^{\pi/2} -\cos^4(t)\sin(t) + \sin^4(t)\cos(t) dt \\
&= 3 \int_0^{\pi/2} -\cos^4(t)\sin(t) + \sin^4(t)\cos(t) dt \\
&= \left(-3 \int_0^{\pi/2} \cos^4(t)\sin(t) dt \right) + \left(3 \int_0^{\pi/2} \sin^4(t)\cos(t) dt \right)
\end{aligned}$$

To evaluate these last two integrals, use the u-substitution $u = \cos(t)$ on the left and $u = \sin(t)$ on the right:

$$\begin{aligned}
\int_C F dC &= \left(3 \int_1^{0} u^4 dt \right) + \left(3 \int_0^{1} u^4 dt \right) \\
&= \left(-3 \int_0^{1} u^4 dt \right) + \left(3 \int_0^{1} u^4 dt \right) \\
&= 0.
\end{aligned}$$

Alternately, for a much faster solution, just use Stokes' theorem, which tells us that the integral of F over C is the integral of $(\nabla \times F) \cdot \mathbf{n}$ over S. Then, because

$$\begin{aligned}
\text{curl}(F) = \nabla \times F &= \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \right) \\
&= (0 - 0, 0 - 0, 0 - 0) \\
&= (0, 0, 0),
\end{aligned}$$

we know that $(\nabla \times F) \cdot \mathbf{n}$ is identically 0, and thus that the integral of this quantity over S is also zero.