Math 1c	TA: Padraic Bartlett
Recitation 8: Change of Variables	
Week 8	*Caltech 2011*

1 Random Question

1. Show that the 5-dimensional unit ball $B_5 = \{\mathbf{x} \in \mathbb{R}^5 : ||x|| \leq 1\}$ has volume $8\pi^2/15$.

2. Show that this volume is the largest volume attained by the n-dimensional unit spheres: i.e. show that for any $B_n = \{\mathbf{x} \in \mathbb{R}^n : ||x|| \leq 1\}$, vol$(B_n)$ <vol(B_5).

So, in some sense, the 5-dimensional sphere takes up "more room" than any of the other spheres in their respective dimensions. Strange, right? We'll actually sketch out how to prove this at the end of the lecture, so read on if you're curious! (and skip the end if you don't want it spoiled for you!)

2 Change of Variables

2.1 Change of variables: the theorem and motivation.

(Here, we discuss a little bit of "why" our change of variables formula is what it is. Feel free to skip to the statement of the theorem at the end of this subsection, if you'd rather hurry up and get to the examples!)

The concept of "changing variables" is one we've ran into in single-variable calculus:

Theorem 1 *(Change of variables, single-variable form) Suppose that f is a continuous function over the interval $(g(a), g(b))$ and that g is a C^1 map from (a, b) to $(g(a), g(b))$. Then, we have that*

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x)) \cdot g'(x)dx.$$

The idea here, roughly, was the following: the integral of f over the interval $(g(a), g(b))$ is the same as the integral of $f \circ g$ over the interval (a, b), as long as we correct for how g "distorts space. In other words, on the left (where g's been applied to the domain (a, b)), we're integrating with respect to dx, the change in x: however, on the right, we're now integrating $f \circ g$, and therefore we should integrate with respect to $d(g(x)) = g'(x)dx$.

So: in multiple variables, we want to have a similar theorem! Basically, given a continuous function $f : \mathbb{R}^n \to \mathbb{R}$, a domain $R \subset \mathbb{R}^n$, and a differentiable map $g : \mathbb{R}^n \to \mathbb{R}^n$, we want a way to relate the integral of f over $g(R)$ and the integral of $f \circ g$ over R.

How can we do this? In other words, how can we correct for how g "distorts space," like we did for our single-variable case? Well: locally, we know that small changes in the vector $\mathbf{x}$ are measured by $D(g(\mathbf{x}))$, the $n \times n$ matrix of partial derivatives of g. Specifically, from

Math 1b, we know that $\det(D(g(\mathbf{x})))$ measures the volume of the image of the unit cube under the map $D(g(\mathbf{x}))$. So, in a sense, this quantity – $\det(D(g(\mathbf{x})))$, the determinant of the Jacobian of g – is telling us how much g is inflating or shrinking space by at the point $\mathbf{x}$! So, we might hope that this is the correct quantity to scale by. As it turns out, it is! Specifically, we have the following theorem:

Theorem 2 *(Change of variables, multiple-variable form) Suppose that R is an open region in $\mathbb{R}^n$, g is a C^1 map $\mathbb{R}^n \to \mathbb{R}^n$ on an open neighborhood of R, and that f is a continuous function on an open neighborhood of the region $g(R)$. Then, we have*

$$\int_{g(R)} f(\mathbf{x})dV = \int_R f(g(\mathbf{x})) \cdot g'(\mathbf{x})dV.$$

2.2 Common variable changes.

There are three exceptionally common changes of variable, which we review here briefly:

Theorem 3 *(Change of variables, polar:) Let $\gamma : [0,\infty) \times [0, 2\pi)$ be the polar coördinates map $(r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$. Clearly, γ is C^∞. Then $D(\gamma(r,\theta)) = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$, $\det(D(\gamma(r,\theta))) = r$, and we have*

$$\int_{\gamma(R)} f(x, y)dV = \int_R f(r\cos(\theta), r\sin(\theta)) \cdot rdV,$$

for any region R in $\mathbb{R}^2$ and any continuous function f on an open neigborhood of R.

In other words, if we have a region R described by polar coördinates, we can say that the integral of f over $\gamma(R)$ is just the integral of $r \cdot f(r\cos(\theta), r\sin(\theta))$ over this region interpreted in Euclidean coördinates. For example, suppose that R was the unit disk, which we can express using our polar coördinates map as $\gamma([0, 1] \times [0, 2\pi))$. Then, change of variables tells us that the integral of f over the unit disk is just the integral of $r \cdot f(r\cos(\theta), r\sin(\theta))$ over the Euclidean-coördinates rectangle $[0, 1] \times [0, 2\pi)$.

Cylindrical coördinates are similar:

Theorem 4 *(Change of variables, cylindrical:) Let $\gamma : [0,\infty) \times [0, 2\pi) \times \mathbb{R}$ be the cylindrical coördinates map $(r, \theta, z) \mapsto (r\cos(\theta), r\sin(\theta), z)$. Clearly, γ is C^∞. Then $D(\gamma(r,\theta)) = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\det(D(\gamma(r,\theta))) = r$, and we have*

$$\int_{\gamma(R)} f(x, y)dV = \int_R f(r\cos(\theta), r\sin(\theta), z) \cdot rdV,$$

for any region R in $\mathbb{R}^3$ and any continuous function f on an open neigborhood of R.

Spherical coördinates are a bit trickier, but have a similar form:

Theorem 5 *(Change of variables, spherical:) Let* $\gamma : [0,\infty)\times[0,\pi)\times[0,2\pi)$ *be the cylindrical coördinates map* $(r,\varphi,\theta)\mapsto(r\cos(\varphi), r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta))$. *Clearly,* γ *is* C^∞. *Then* $D(\gamma(r,\theta)) = \begin{bmatrix} \cos(\varphi) & -r\sin(\varphi) & 0 \\ \sin(\varphi)\cos(\theta) & r\cos(\varphi)\cos(\theta) & -r\sin(\varphi)\sin(\theta) \\ \sin(\varphi)\sin(\theta) & r\cos(\varphi)\sin(\theta) & r\sin(\varphi)\cos(\theta) \end{bmatrix}$, $\det(D(\gamma(r,\theta))) = r^2\sin(\varphi)$, *and we have*

$$\int_{\gamma(R)} f(x,y)dV = \int_R f(r\cos(\varphi), r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta))\cdot r^2\sin(\varphi)dV,$$

for any region R *in* $\mathbb{R}^3$ *and any continuous function* f *on an open neigborhood of* R.

There are a few other coördinate transforms that will often come up:

- Various translations of space: i.e. maps $(x,y,z)\mapsto(x+c_1, y+c_2, z+c_3)$. The determinant of the Jacobian of such maps will always be 1.
- Various ways to scale space: i.e. maps $(x,y)\mapsto(\lambda_1 x, \lambda_2 y)$. The determinant of the Jacobian of such maps will be the product of these scaling constants $\lambda_1\cdot\ldots\cdot\lambda_n$.
- Various compositions of these maps: i.e. a translation map, followed by a spherical coördinates map, followed by a scaling map. Using the chain rule, the determinant of the Jacobian of any such composition of maps is just the product of the determinants of the individual Jacobians.

Often, the trickiest thing to do in a problem is realize which coördinate system makes the most sense to use, and then to use it. We work some examples below, ranging from trivial to deeply complex:

2.3 Examples.

Example. Find the area in $\mathbb{R}^2$ of the ellipse $E = \{(x,y) : x^2/a^2 + y^2/b^2 = 1\}$, using change of variables.

Solution. We are looking for the integral of the function 1 over the set E. To find this, first notice that our ellipse E is just the image of the unit disk D under the scaling map $\gamma(x,y) = (ax, by)$. Therefore, via one application of change of variables, we have

$$\int_{\gamma(D)} 1\ dV = \int_D 1\cdot\det\begin{bmatrix} a & 0 \\ 0 & b\end{bmatrix} dV = \int_D ab\ dV.$$

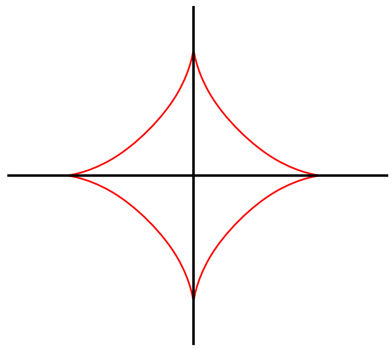
Now, using polar coördinates to describe the unit disk D as the image of $[0,1]\times[0,2\pi]$ under the map $\alpha(r,\theta)\to(r\cos(\theta), r\sin(\theta))$, we can apply change of variables again to get that

$$\int_{\gamma(D)} 1\ dV = \int_{[0,1]\times[0,2\pi]} 1\cdot\det(D\alpha)dV = \int_0^1\int_0^{2\pi} ab\cdot r\ d\theta dr = \pi ab.$$

So the area of our ellipse is πab.

(Note that we calculated $\det(D\alpha)$ in our earlier discussion of polar coördinate transforms, which is why we just plugged in r for it as opposed to rederiving it again. In your homework sets, when you're using spherical, cylindrical, or polar coördinates, feel free to just use the fact that you know that the determinants of the Jacobians of these maps are! as we've calculated them a thousand times and will do so a thousand more. For anything more complicated, though, please show your work.)

Example. Find the area enclosed by the astroid curve $\{(x, y) : x^{2/3} + x^{2/3} = 1\}$.



Solution. So: we're trying to find the 2-dimensional volume of the set $A = \{(x, y) : x^{2/3} + x^{2/3} \leq 1\}$. How can we do this? Well: polar coördinates seem tempting, but as it turns out a direct application of them won't work too well here – the set A is actually really difficult to express in polar coördinates! The **idea** behind polar coördinates, however, seems quite promising!

I.e.: we used polar coördinates $(\cos(\theta), \sin(\theta))$ to describe the set of points $\{(x, y) : x^2 + y^2 = 1\}$. So, to describe the set $\{(x, y) : x^{2/3} + x^{2/3} = 1\}$, it's natural to want to use the parametrization $(\cos^3(\theta), \sin^3(\theta))$! More generally, we can express the set A as the image of the rectangle $[0, 1] \times [0, 2\pi]$ under the map

$$\gamma(r, \theta) = (r \cos^3(\theta), r \sin^3(\theta)).$$

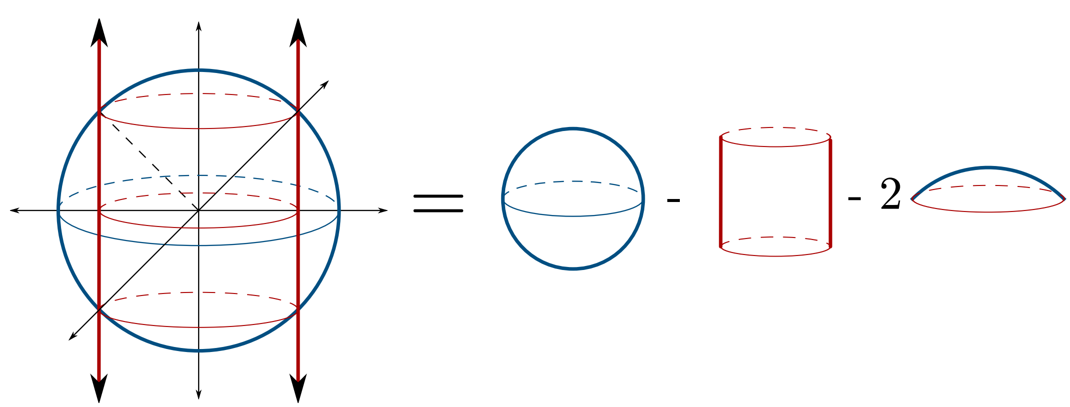
So: if we use change of variables with this new, polar-like map, we have that

$$\begin{aligned}
\int_A 1\ dV &= \int_{[0,1]\times[0,2\pi]} 1 \cdot \det\left(\begin{bmatrix} \cos^3(\theta) & -3r\cos^2(\theta)\sin(\theta) \\ \sin^3(\theta) & 3r\sin^2(\theta)\cos(\theta) \end{bmatrix}\right) dV \\
&= \int_0^1 \int_0^{2\pi} 3r\left(\cos^4(\theta)\sin^2(\theta) + \sin^4(\theta)\cos^2(\theta)\right) d\theta dr \\
&= \int_0^1 \int_0^{2\pi} 3r\cos^2(\theta)\sin^2(\theta)\left(\cos^2(\theta) + \sin^2(\theta)\right) d\theta dr \\
&= \int_0^1 \int_0^{2\pi} 3r\cos^2(\theta)\sin^2(\theta) d\theta dr \\
&= \int_0^1 \int_0^{2\pi} 3r\frac{\sin^2(2\theta)}{4} d\theta dr \\
&= \int_0^1 \int_0^{2\pi} 3r\frac{1-\cos(4\theta)}{8} d\theta dr \\
&= \int_0^1 \frac{3r\cdot\pi}{4} dr \\
&= 3\pi/8.
\end{aligned}$$

(The only tricks used in the integral above are the double-angle formulas and the observation that $\cos^2 + \sin^2 = 1$. Do try to remember your trig identities! – they come up in the weirdest places, and can make otherwise awful integrals remarkably easy.)

Example. Let $B_3 = \{(x,y,z) : x^2+y^2+z^2 \leq 1\}$ be the unit ball and $C = \{(x,y,z) : x^2+y^2 \leq 1/4\}$ be a cylinder of radius $1/2$ around the z-axis. Find the volume of the set of points $R = B_3 \setminus C$: i.e. the collection R of points both contained within the unit ball B_3 and outside of the cylinder C.

Solution. So: to set this problem up, let's first draw a picture of the situation:



We want to find the volume trapped between the cylinder and the sphere. To do this, we can simply take the volume of the unit sphere and subtract off the collection of points trapped within both the sphere in the cylinder. We can decompose those points into two kinds:

- The points trapped within the red “can” $\{(x,y,z) : x^2+y^2 \le 1/4, -\sqrt{3}/2 \le z \le \sqrt{3}/2\}$.
- The points trapped within one of the two red and blue ”sphere caps” $\{(x,y,z) : x^2+y^2+z^2 \le 1, z > \sqrt{3}/2\}$ or $\{(x,y,z) : x^2+y^2+z^2 \le 1, z < -\sqrt{3}/2\}$.

To find the volume of R, then, it suffices to find these three volumes, and then subtract them off of each other.

For fun, we start by rederiving the formula for the area of a sphere, using spherical coördinates:

$$\begin{aligned}
\int_{B_3} 1dV &= \int_{[0,1]\times[0,\pi)\times[0,2\pi)} 1\cdot r^2\sin(\varphi)dV \\
&= \int_0^1\int_0^\pi\int_0^{2\pi} r^2\sin(\varphi)d\theta d\varphi dr \\
&= \int_0^1\int_0^\pi 2\pi r^2\sin(\varphi)d\varphi dr \\
&= \int_0^1 \left(2\pi r^2\cos(\varphi)\right)\Big|_0^\pi dr \\
&= \int_0^1 4\pi r^2 dr \\
&= \frac{4\pi}{3}.
\end{aligned}$$

Now, we use cylindrical coördinates to find the volume of our “can”:

$$\begin{aligned}
\int_{\text{“can”}} 1dV &= \int_{[0,1/2]\times[0,2\pi)\times[-\sqrt{3}/2,\sqrt{3}/2]} 1\cdot r\ dV \\
&= \int_0^{1/2}\int_0^{2\pi}\int_{-\sqrt{3}/2}^{\sqrt{3}/2} r\ dzd\theta dr \\
&= \int_0^{1/2}\int_0^{2\pi} r\sqrt{3}\ d\theta dr \\
&= \int_0^{1/2} 2r\pi\sqrt{3}dr \\
&= \frac{\pi\sqrt{3}}{4}.
\end{aligned}$$

Finally, we need to find the volume of our sphere caps. You could attempt to use spherical coördinates to describe these sphere caps, but the equations for what constraints we would have on r will involve some somewhat painful trig equations. However, if we use cylindrical coördinates, we can see that the northern sphere cap can be described as the set of points $\{(r,\theta,z) : 0 \le r \le 1/2, \sqrt{3}/2 \le z \le \sqrt{1-r^2}\}$. This is because the points in our sphere cap are those with (x,y) part inside of our cylinder, and z- coördinate between the floor $z = \sqrt{3}/2$ of the sphere cap and the ceiling $z^2+r^2 = 1$ of the sphere cap.

So, this is not such a bad set! Specifically, if we use this description of the sphere cap and apply change of variables, we have

$$\begin{aligned}
\int_{\text{“sphere cap”}} 1dV &= \int_0^{1/2}\int_0^{2\pi}\int_{\sqrt{3}/2}^{\sqrt{1-r^2}} r\ dz d\theta dr \\
&= \int_0^{1/2}\int_0^{2\pi} \left(\sqrt{1-r^2}-\sqrt{3}/2\right) r\ d\theta dr \\
&= \int_0^{1/2} 2\pi \left(\sqrt{1-r^2}-\sqrt{3}/2\right) r\ dr \\
&= \left(\int_0^{1/2} 2\pi r\sqrt{1-r^2}\ dr\right) - \left(\int_0^{1/2} \pi r\sqrt{3} dr\right) \\
&= \left(\int_0^{1/2} 2\pi r\sqrt{1-r^2}\ dr\right) - \frac{\pi\sqrt{3}}{8} \\
&= \left(\int_1^{3/4} -\pi\sqrt{u}\ du\right) - \frac{\pi\sqrt{3}}{8} \\
&= \left(\int_{3/4}^{1} \pi\sqrt{u}\ du\right) - \frac{\pi\sqrt{3}}{8} \\
&= \left(\frac{2\pi}{3}u^{3/2}\right)\Bigg|_{3/4}^{1} - \frac{\pi\sqrt{3}}{8} \\
&= \frac{2\pi}{3} - \frac{\pi\sqrt{3}}{4} - \frac{\pi\sqrt{3}}{8} \\
&= \frac{2\pi}{3} - \frac{3\pi\sqrt{3}}{8}
\end{aligned}$$

where we used the substitution $u = 1-r^2, du = -2r$ to evaluate the first of our two integrals.

So, in summary, the volume of our set R is

$$\begin{aligned}
&\frac{4\pi}{3} - \frac{\pi\sqrt{3}}{4} - 2\left(\frac{2\pi}{3} - \frac{3\pi\sqrt{3}}{8}\right) \\
=&\frac{\pi\sqrt{3}}{8}
\end{aligned}$$

(up to any algebraic mistakes we may have made along the way.)

2.4 An outline for how to deal with the random question

As it turns out, you can use change of variables to prove the remarkably surprising result described in the random question! Specifically, notice that we can “generalize” our three-dimensional spherical coördinates to “n-dimensional” spherical coördinates! In other words, let $r \in [0, \infty)$, $\varphi_1, \ldots \varphi_{n-2} \in [0, \pi)$, and $\theta \in [0, 2\pi)$. Now, consider the map γ that sends a point $(r, \varphi_1, \ldots \varphi_{n-2}, \theta)$ to a point in $\mathbb{R}^n$ with the following coördinates:

$$\begin{aligned}
x_1 &= r\cos(\varphi_1), \\
x_2 &= r\sin(\varphi_1)\cos(\varphi_2) \\
x_3 &= r\sin(\varphi_1)\sin(\varphi_2)\cos(\varphi_3) \\
&\vdots \\
x_{n-2} &= r\sin(\varphi_1)\cdots\sin(\varphi_{n-3})\cos(\varphi_{n-2}) \\
x_{n-1} &= r\sin(\varphi_1)\cdots\sin(\varphi_{n-2})\sin(\varphi_{n-2})\cos(\theta) \\
x_n &= r\sin(\varphi_1)\cdots\sin(\varphi_{n-2})\sin(\varphi_{n-2})\sin(\theta)
\end{aligned}$$

You can show that this point $\mathbf{x}$ is a point distance r from the origin that has angle φ_i with the first $n-2$ coördinate axes and angle θ with the $(n-1)$-th axis. In other words, this is really a n-dimensional generalization of the concept of spherical coördinates.

Inductively, you can show that $\det(D(\gamma))$ has the form

$$r^{n-1}\sin^{n-2}(\varphi_1)\sin^{n-3}(\varphi_2)\cdots\sin(\varphi_{n-2}),$$

and thus that the volume of the n-dimensional ball, via change of variables, is just the integral

$$\int_0^1\int_0^\pi \ldots \int_0^\pi\int_0^{2\pi} r^{n-1}\sin^{n-2}(\varphi_1)\sin^{n-3}(\varphi_2)\cdots\sin(\varphi_{n-2})d\varphi_1 d\varphi_2\cdots d\varphi_{n-1}dr.$$

From there, you can use induction to prove the recursion relation

$$\text{vol}(B_n) = \frac{2\pi}{n}\text{vol}(B_{n-2}),$$

which tells us that for $n \geq 7$, $\text{vol}(B_n)$ is strictly smaller than $\text{vol}(B_{n-2})$. Checking the volumes for the balls $B_1, \ldots B_6$ then shows that $\text{vol}(B_5)$ is the greatest amongst those six balls: therefore, the volume of the five-dimensional unit ball B_5 is greater than the n-dimensional volume of **any** of the other n-dimensional unit balls, because the volumes are (as shown) decreasing for $n > 6$!

None of these calculations are beyond you; if you're bored and want some practice with change of variables, feel free to try them! (and feel free to ask me for pointers, if you're stuck!)