Math 1c	TA: Padraic Bartlett

Recitation 4: Lagrange Multipliers and Integration

Week 4	*Caltech 2011*

1 Random Question

Hey! So, this random question is pretty tightly tied to today's lecture and the concept of "content zero:" so, even if you usually skip these, do read this one so that the later bits of the lecture make sense, and understand the first question (the one that asks why the Cantor set is content zero.)

So, as we'll discuss later today, a set A is said to have "content zero" if and only if for any $\epsilon > 0$ there is some finite collection of rectangles $R_1 \ldots R_n$ such that

- $A \subseteq \cup_{i=1}^n R_i$
- $\sum_{i=}^n \text{vol}(R_i) < \epsilon$.

In other words, a set has content zero iff you can squeeze it inside a finite set of rectangles of arbitrarily small volume; i.e. it's not taking up any "space" in $\mathbb{R}^n$, in a sense.

With that in mind, consider the following definition of the Cantor set:

Definition. We define the Cantor set C_∞ in stages, as follows:

- $C_0 = [0, 1]$.
- $C_1 = C_0$ with its middle-third removed: i.e. $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.
- $C_2 = C_1$ with the middle-third of each of its intervals removed: i.e. $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.
- $\vdots$
- $C_{n+1} = C_n$ with the middle-third of each of the intervals in C_n removed.
- $\vdots$
- $C_\infty = \bigcap_{n=1}^\infty C_n$.

Prove the following claims about the Cantor set:

1. Show that each set C_n has length $= 2^n/3^n$. Use this to prove that C_∞ is content zero.
2. Show that the interior of the Cantor set is the empty set – i.e. show that the Cantor set contains no open interval of any length.
3. Despite the above statement, show that the Cantor set has, in a sense, "as many" points as $[0, 1]$: i.e. that you can find a bijection from C_∞ to $[0, 1]$. (Hint: what do elements of C_∞ look like when you write them in ternary?)

4. Define the following "Cantor-like" set as follows:

   - $C_0 = [0, 1]$.
   - $C_1 = C_0$ with its middle-quarter removed.
   - $C_2 = C_1$ with the middle-$(\frac{1}{8} \cdot \frac{4}{3})$ of each of its intervals removed.
   - $C_3 = C_2$ with the middle-$(\frac{1}{16} \cdot \frac{8}{7})$ of each of its intervals removed.
   - $\vdots$
   - $C_{n+1} = C_n$ with the middle-$(\frac{1}{2^{n+1}} \cdot \frac{2^n}{2^n - 1})$ of each of the intervals in C_n removed.
   - $\vdots$
   - $C_\infty = \bigcap_{n=1}^{\infty} C_n$.

   Show that the interior of this set is also empty. Show, however, that in some certain well-defined way, this set has "length" $1/2$.

Today's lecture is insanely long, because we have an absurd amount of material to cover; as a result, in Thursday's recitation we didn't get to make it through a large amount of the integration section. I've tried to pick the problems there so that they will be helpful to think about when attacking your HW, so do look there when you're working on your problems!

2 Lagrange Multipliers

2.1 Statement of the method.

In last week's recitation, we talked about how to use the derivatives to find extremal points of a functions on some region D. However, in real life, we often won't be only attempting to maximize a function on some nice open rectangle D; we'll often have functions we want to maximize given some set of weird constraints, like restricting its points to the surface of the unit ball (i.e. $x_1^2 + \ldots + x_n^2 = 1$) or making sure that the first and third coördinates are reciprocals of each other, or some sort of odd conditions like that.

How can we find maxima on such strange sets, that don't have interiors? As it turns out, the method of Lagrange multipliers is precisely the tool that does that for us. We state the method here:

Proposition 1 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a function whose extremal values $\{\mathbf{x}\}$ we would like to find, given the constraints $g_1(\mathbf{x}) = c_1, \ldots g_m(\mathbf{x}) = c_m$, for some collection of functions $g_1, \ldots g_m : \mathbb{R}^n \to \mathbb{R}$. Then, we have the following result: if $\mathbf{a}$ is an extremal value of f restricted to the set $S = \{\mathbf{x} : \forall i, g_i(\mathbf{x}) = c_i\}$, and all of the gradients $\nabla g_1(\mathbf{a}), \ldots \nabla g_m(\mathbf{a})$ are linearly independent, then $\nabla f(\mathbf{a})$ is linearly dependent on the $\nabla g_i(\mathbf{a})$'s. In other words, there are constants λ_i such that*

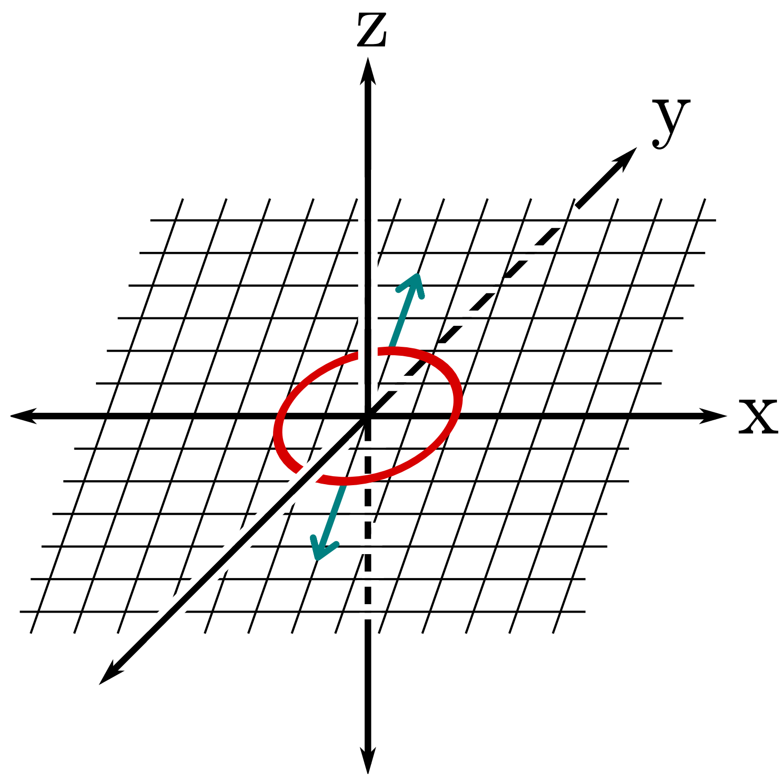
$$\nabla f(\mathbf{a}) = \lambda_1 \cdot \nabla g_1(\mathbf{a}) + \ldots \lambda_m \cdot \nabla g_m(\mathbf{a}).$$

2.2 A visual example.

Here's a visual example, to illustrate what's going on:

Question 2 *Suppose that we want to maximize the function* $f(x, y) = -y$ *on the set of constraints* $g(x, y) = x^2 + y^2 = 1$*. What values of* x *and* y *do this for us?*

Solution. So, before we start, we graph the situation at hand, so we can visually see what's going on:



Visually, we can see that the maxima of our function $f(x, y) = -y$ (the plane drawn as a grid in the above picture) on the set $g(x, y) = 1$ (the red circle in the above picture) occur at the points $(0, 1, -1)$ and $(0, -1, 1)$; there, we can "see" visually that the gradients of these two functions are linear multiples of each other, as the gradient of g there is $(0, \pm 2)$, and the gradient of f is $(0, 1)$.

Indeed: if we check all of the possible points (x, y) and constants λ where $\nabla(f)(x, y) = \lambda\nabla(g)(x, y)$, we can see that

$$\begin{aligned} \nabla(f)(x, y) = (0, 1) = \nabla(g)(x, y) = (2x, 2x) \\ \Rightarrow 0 = \lambda \cdot 2x, 1 = \lambda \cdot 2y \\ \Rightarrow \lambda \neq 0 \\ \Rightarrow x = 0, y = \pm 1, \lambda = \pm y \end{aligned}$$

and thus that the only extremal points of our function f on this set are $(x, y) = (0, \pm 1)$. (We are justified in claiming that these are the *only* extremal points because the method of Lagrange multipliers detects extrema whenever the set of ∇g_i's are all linearly independent: so, because $\nabla g(x, y) = (2x, 2y)$ is never (0,0) on the unit circle, this set of "one" ∇g is linearly independent, and therefore the Lagrange multipler method will find all of our extrema.)

2.3 A more detailed example.

The above is a kind-of trivial example of when we use Lagrange multipliers: when the f and g are as clearly spelled out as above, it's usually not too hard to apply our method. In many problems, it's usually a bit more ambiguous than this; the next example provides more of a "real-world" use of the method of Lagrange multipliers, wherein we prove a famous inequality:

Question 3 *Can you prove that the arithmetic mean of a collection of positive numbers is always greater than or equal to their geometric mean, using the method of Lagrange multipliers? In other words, can you prove that for any* $x_1, \ldots x_n \geq 0$*, we have*

$$\frac{x_1 + \ldots x_n}{n} \geq (x_1 \cdot \ldots \cdot x_n)^{1/n}?$$

Solution. So: how can we prove such a statement with the method of Lagrange multipliers? Well: let's think about what we're trying to prove. We want the claim

$$\frac{x_1 + \ldots x_n}{n} \geq (x_1 \cdot \ldots \cdot x_n)^{1/n}$$

to hold for any vector $\mathbf{x}$ with all positive coördinates. So: to make things easier on ourselves, suppose that we're specifically looking at a vector $\mathbf{x}$ with arithmetic mean $\frac{x_1 + \ldots x_n}{n} = c$; then, we're trying to prove that

- $(x_1 \cdot \ldots \cdot x_n)^{1/n}$ is bounded above, by this constant c, whenever
- the vector $\mathbf{x}$ is subject to the constraint $\frac{x_1 + \ldots x_n}{n} = c$!

So: if we can simply find the **maximum** of $(x_1 \cdot \ldots \cdot x_n)^{1/n}$ under this constraint $\frac{x_1 + \ldots x_n}{n} = c$, we will know whether or not our inequality holds – because if the maximum is $\leq c$, then surely all of the values of our function must be $\leq c$! To phrase this in the language of Lagrange multipliers, we have

- $f(\mathbf{x}) = (x_1 \cdot \ldots \cdot x_n)^{1/n}$,
- $g(\mathbf{x}) = \frac{x_1 + \ldots x_n}{n}$, constrained to equal some c,
- and we want to find the maximum of f under the constraint $g(\mathbf{x}) = c$.

With this setup, we can note that since $\nabla g = (\frac{1}{n}, \dots \frac{1}{n})$ is never the zero vector, it constitutes a linearly independent set of one vector, and thus that the only extremal values of f under our constraint will occur at points where $\nabla f = \lambda \cdot \nabla g$: i.e. $\mathbf{x}$ such that

$$\begin{aligned}
\left(\frac{1}{n}, \frac{1}{n}, \dots\right) &= \left(\frac{\partial}{\partial x_1}\left((x_1)^{1/n} \cdot (x_2 \cdot \dots x_n)^{1/n}\right), \frac{\partial}{\partial x_2}\left((x_2)^{1/n} \cdot (x_1 \cdot x_3 \cdot \dots x_n)^{1/n}\right), \dots\right) \\
&= \left(\frac{1}{n}\left((x_1)^{1/n-1} \cdot (x_2 \cdot \dots x_n)^{1/n}\right), \frac{1}{n}\left((x_2)^{1/n-1} \cdot (x_1 \cdot x_3 \cdot \dots x_n)^{1/n}\right), \dots\right) \\
\Rightarrow \frac{1}{n} &= \frac{1}{n}\left((x_1)^{1/n-1} \cdot (x_2 \cdot \dots x_n)^{1/n}\right), \\
\frac{1}{n} &= \frac{1}{n}\left((x_2)^{1/n-1} \cdot (x_1 \cdot x_3 \cdot \dots x_n)^{1/n}\right), \\
&\vdots \\
\Rightarrow x_1 &= x_1^{1/n} \cdot (x_2 \cdot \dots x_n)^{1/n}, \\
x_2 &= x_2^{1/n} \cdot (x_1 \cdot x_3 \cdot \dots x_n)^{1/n}, \\
&\vdots \\
\Rightarrow x_i &= (x_1 \cdot \dots x_n)^{1/n}, \text{ for every } i.
\end{aligned}$$

So, the only critical point of our function f on the level set $g(\mathbf{x}) = c$ occurs when all of the coördinates x_i are equal. This is clearly a maximum, as it is the only critical point on our entire set $g(\mathbf{x}) = c$, f evaluated at any point on the boundary of $g(\mathbf{x}) = c$ (those where one coördinate x_i is 0) is identically 0, and $0 < f(x_1, \dots x_1) = (x_1^n)^{1/n} = x_1$.

Thus, because $f(x_1, \dots x_1) = x_1 = g(x_1, \dots x_1) = c$, we have that $f(\mathbf{x}) \leq c$ at f's maximum, and thus that $f(\mathbf{x}) \leq c$ for all values of $\mathbf{x}$ such that $g(\mathbf{x}) = c$. In other words, for any constant c, we've proven that if the arithmetic mean of a number is c, its geometric mean must be $\leq c$: which is precisely what we wanted to prove.

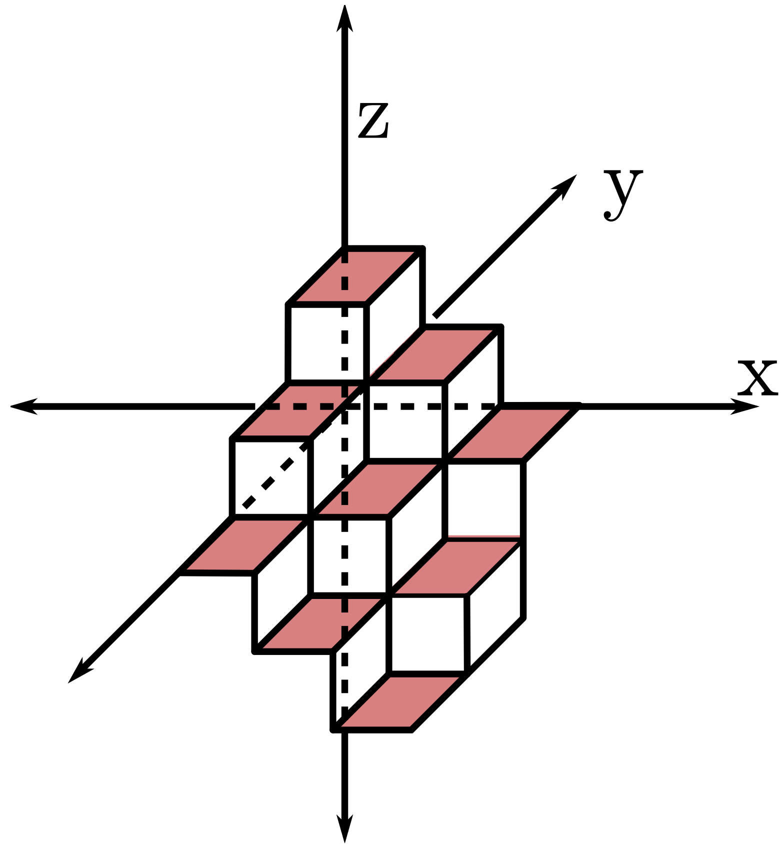
3 Integration of Functions from $\mathbb{R}^2$ to $\mathbb{R}$.

Changing tacks entirely, we now move to discussing integration in higher dimensions. Recall the following definitions and theorems:

3.1 Basic definitions and theorems.

Definition. We say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is a **step function** on some rectangle R iff there is a subdivision of R into smaller rectangles $R_1, \dots R_n$, such that f is constant on each of the R_i's.

Example. The function drawn below is a step function on $[0,3] \times [0,3]$:



Definition. The **integral** of a step function f over some rectangle R is simply the sum of the volumes of all of the boxes formed by the R_i's and f's constant value on these R_i's.

Example. The integral of the function drawn above on $[0,3] \times [0,3]$ is simply $2+1+0+1+0+-1+0+-1+-2=0$.

Definition. A function f is integrable over some rectangle R iff there is some *unique* value L such that for any two step functions s,t on R such that $s(x,y) \le f(x,y) \le t(x,y)$ on all of R, we have

$$\iint_R s(x,y)dA \le L \le \iint_R t(x,y)dA.$$

(Note that there will always be some value of L: what you're always trying to prove whenever you use this definition is that this value of L is *unique*.)

Theorem 4 *Suppose that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is integral over some rectangle $[a,b] \times [c,d]$; suppose, furthermore, that for any $y \in [c,d]$, the single-variable calculus integral*

$$\int_a^b f(x,y)dx$$

exists.

Then we can find $\iint\limits_{[a,b]\times[c,d]} f(x,y)dA$ *by simply integrating first with respect to* x *and then with respect to* y*: in other words, we have*

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)dA = \int_c^d \left(\int_a^b f(x,y)dx \right) dy.$$

Practically speaking, the above theorem is how you're going to calculate most integrals; the definitions are really only used in pathological cases, or where you have a function with a lot of discontinuities. However, to use the above theorem, you have to know ahead of time that your function *is* integrable! The following definition and theorem tells us when this happens:

Definition. A set A is said to have "content zero" if, for any $\epsilon > 0$, there is some finite collection of rectangles $R_1 \dots R_n$ such that

- $A \subseteq \cup_{i=1}^n R_i$
- $\sum_{i=}^n \text{vol}(R_i) < \epsilon$.

In other words, a set has content zero iff you can squeeze it inside a finite set of rectangles of arbitrarily small volume; i.e. it's not taking up any "space" in $\mathbb{R}^n$, in a sense.

Theorem 5 *Let* S *be the set of discontinuities of a function* $f : \mathbb{R}^2 \to \mathbb{R}$ *on some rectangle* R*. If* S *has content zero, then* f *is integrable on* R*.*

As a special case, continuous functions are integrable over any rectangle, as their set of discontinuities is the empty set (which trivially has content zero).

3.2 A series of examples.

We work a series of examples, to give a feel for the material:

Question 6 *Does the integral*

$$\iint\limits_{[0,1]\times[0,1]} \min(x,y)dA$$

exist? What is it?

Solution. So: first, notice that this is a continuous function, and therefore by our above theorem this integral exists.

Then, notice that for any fixed $y \in [0,1]$, we have

$$\begin{aligned}
\int_0^1 \min(x,y)dx &= \int_0^y \min(x,y)dx + \int_y^1 \min(x,y)dx \\
&= \int_0^y xdx + \int_y^1 ydx \\
&= \frac{x^2}{2}\Big|_0^y + xy\Big|_y^1 \\
&= \frac{y^2}{2} + y - y^2 \\
&= y - \frac{y^2}{2},
\end{aligned}$$

and therefore that the integral $\int_0^1 \min(x,y)dx$ exists for all y. Then, by our earlier theorem, we can evaluate $\iint\limits_{[0,1]\times[0,1]} \min(x,y)dA$ by simply integrating with respect to x and then with respect to y:

$$\begin{aligned}
\iint\limits_{[0,1]\times[0,1]} \min(x,y)dA &= \int_0^1 \int_0^1 \min(x,y)dxdy \\
&= \int_0^1 \left(y - \frac{y^2}{2}\right) dy \\
&= \frac{y^2}{2} - \frac{y^3}{6}\Big|_0^1 \\
&= \frac{1}{3}.
\end{aligned}$$

So, if you take two random numbers in $[0,1]$ and look at the smaller of the two, the average value you'd see is $1/3$.

Question 7 *Suppose that $f(x,y)$ is the function equal to 1 whenever x and y are both in the Cantor set C_∞, and 0 otherwise. Does the integral*

$$\iint\limits_{[0,1]\times[0,1]} f(x,y)dA$$

exist? What is it?

Solution. So: notice first that $C_\infty \times C_\infty$ is contained within each the following sets:

- $C_0 \times C_0 = [0,1]^2$.
- $C_1 \times C_1 = [0,\frac{1}{3}]^2 \cup ([0,\frac{1}{3}] \times [\frac{2}{3},1]) \cup ([\frac{2}{3},1] \times [0,\frac{1}{3}]) \cup [\frac{2}{3},1]^2$
- $C_2 \times C_2 = [0,\frac{1}{9}]^2 \cup [0,\frac{1}{9}] \times [\frac{2}{9},\frac{1}{3}] \cup \ldots$

- $\vdots$

and that for any $C_n \times C_n$, we have 2^{2n} parts, each with volume $1/3^{2n}$: therefore, each of these sets has volume $(\frac{2}{3})^{2n}$, which goes to 0 as n gets large.

As well, we know that our function f is continuous on the complement of each of the $C_n \times C_n$'s, as

- each of these $C_n \times C_n$'s are closed sets, so
- their complement is open, and therefore
- we can find an open ball around any point in the complement that has no points in $C_\infty \times C_\infty$, on which
- our function is identically 0 and therefore continuous at that point.

So our function is continuous on the complement of each $C_n \times C_n$; this means that our function is discontinuous, at worst, at the points that are in every $C_n \times C_n$: i.e. our function is discontinuous at worst on the set $C_\infty \times C_\infty$.

But this set is of content zero, as it's contained within each of the $C_n \times C_n$'s! So our function is integrable. Furthermore, we know that if

- s is the step function identically equal to 0, and
- t_n is the step function equal to 1 on $C_n \times C_n$, and 0 otherwise,

we have that $s(x,y) \leq f(x,y) \leq t_n(x,y)$ for all n, and also

$$\iint_{[0,1]\times[0,1]} s(x,y)dA = 0, \quad \iint_{[0,1]\times[0,1]} t_n(x,y)dA = \left(\frac{2}{3}\right)^{2n}$$

the right-hand part of which converges to 0 as $n \to \infty$. Therefore, the only number between these two step functions for all n is 0.

Therefore, by definition, the integral of this function is 0.

Question 8 *Suppose that $f(x,y)$ is the function equal to $\frac{1}{qs}$ whenever $x = \frac{p}{q}$ and $y = \frac{r}{s}$ are both rational numbers in written in lowest terms, and 0 otherwise. Does the integral*

$$\iint_{[0,n]\times[0,n]} f(x,y)dA$$

exist? What is it?

Solution. So: where is this function discontinuous? Well: take any point (x,y) where one of $x, y \notin \mathbb{Q}$. Then, $f(x,y) = 0$. What happens for points in the plane that are very close to (x,y)? Well, if it's x that's irrational, we claim that any value x' very close to x will have to either be irrational or have very large denominator.

Why is that? Well, pick n large enough that the n-th decimal place of x disagrees with the $n-1$-th decimal place of x. Then, if x' is within $10^{-(n+1)}$ of x, we must have that x

and x' agree on their $n-1$ first decimal places. (This is because if the n-th decimal place and the $n-1$th decimal place of x disagree, then specifically they're not both 9 or both 0: therefore, both $x - 10^{-(n+1)}$ and $x + 10^{-(n+1)}$ agree on their first $n-1$ decimal places, as you're at most adding or subtracting 1 from the $n+1$-th decimal place. The same is therefore true for any number within $10^{-(n+1)}$ of x.)

But x is irrational, and irrational numbers cannot be expressed as rational numbers: therefore, for very large values of n, in order to agree with the first $n-1$ digits of x, our number x' will either have to be irrational or be the ratio of ever-increasingly-large numbers. Therefore, the denominatory grows arbitrarily large!

Upshot of this: if we take a ball of radius $10^{-(n+1)}$ around the point (x, y) where one of these coördinates is irrational, and let $n \to \infty$, the values of f taken on in this ball go to 0. Therefore, our function is **continuous** at any such point (x, y) where one (or both) of x, y are irrational!

However, where both x, y are rational, our function is equal to some fixed constant $\frac{1}{qs} \neq 0$. In any ball around (x, y), we know (from HW #1) that there are points with irrational coördinates in that ball, and therefore points on which our function is 0; therefore, because there are points arbitrarily close to (x, y) where our function is 0, and $f(x, y) \neq 0$, we know our function is **discontinuous** at all points $(x, y) \in \mathbb{Q}^2 \cap [0, 1]^2$.

So, this tells us entirely where our function is continuous and discontinuous! As such, you might be tempted to try to show that this set of discontinuities is content zero, to use our earlier theorems. **IT IS NOT.** The set $\mathbb{Q}^2 \cap [0, 1]^2$ is not of content zero: any *finite* collection of rectangles that covers it has to have total volume 1! (If you allow an infinite collection of rectangles, you can make the volume arbitrarily small; this is a notion of **measure**, which we will talk about later.)

Thus, the only tool we have at hand is resorting to the definition to find this function's integral. However, this is not too awful! In specific, do the following:

- Let $\{(p_1, q_1), \ldots (p_m, q_m)\}$ be the collection of all rational pairs in $[0, 1]^2$ where the denominators of both rational numbers are $\leq n$. There are clearly finitely many such pairs.
- Around each point (p_i, q_i), put a rectangle R_i with area $\epsilon/2^i$.
- Let t_n be the step function that is 1 on each such rectangle, and $1/n^2$ otherwise.
- Then, at every point not in such a rectangle, we have $f(x, y) \leq 1/n^2 = t_n(x, y)$, because the denominators of x and y are both $\geq n$ (or one of x, y are irrational); as well, at any point in such a rectangle, $f(x, y) \leq 1 - t_n(x, y)$, trivially.
- As well, we have that $f(x, y) \geq 0$, the step function that's identically 0, trivially.

Then, because $\iint 0 = 0$ and

$$\begin{aligned}
\iint\limits_{[0,1]\times[0,1]} t_n(x,y)dA &= \iint\limits_{R_1\cup\ldots\cup R_m} t_n(x,y)dA + \iint\limits_{\text{everything else}} t_n(x,y)dA \\
&= \iint\limits_{R_1\cup\ldots\cup R_m} 1dA + \iint\limits_{\text{everything else}} \frac{1}{n^2}dA \\
&\leq 1 \cdot \left(\sum_{i=1}^{m} \text{vol}(R_i)\right) + \frac{1}{n^2} \cdot \text{vol}([0,1]^2) \\
&= 1 \cdot \left(\sum_{i=1}^{m} \frac{\epsilon}{2^i}\right) + \frac{1}{n^2} \\
&\leq 1 \cdot \left(\sum_{i=1}^{\infty} \frac{\epsilon}{2^i}\right) + \frac{1}{n^2} \\
&= \epsilon + \frac{1}{n^2}
\end{aligned}$$

we know that the integral of t_n can be made as close to 0 as we want it to be, by picking arbitrarily small ϵ and letting $n \to \infty$. Therefore, the only value between these two integrals is 0, and thus the integral of this function exists and is 0.