STOKE'S THEOREM / CONSERVATIVE VECTOR FIELDS / GAUSS'S THEOREM

TA: PADRAIC BARTLETT

1. Random Question

Question 1.1. Tiling. *Can you:*

1. *cover the plane with disjoint circles?*
2. *cover the plane with disjoint "T"-shapes?*
3. *cover the plane with disks, all of which are either disjoint or touch at at most one point?*
4. *cover $\mathbb{R}^3$ with disjoint circles?*
5. *cover $\mathbb{R}^3$ with disjoint "T"-shapes?*
6. *cover $\mathbb{R}^3$ with disks, all of which are either disjoint or touch at at most one point?*

2. Last Week's HW

Average was, again, about 80/90. Not much really stood out, at least in this section; contact me if you have any specific concerns about grading.

3. Surfaces

So: throughout this course, we've made many many references to surfaces and to the boundaries of surfaces. However, we've never really properly defined what these things are! We rectify this oversight below:

Definition 3.1. A map $f : U \to V$ is called a **homeomorphism** if it is 1-1, onto, and continuous with a well-defined continuous inverse function.

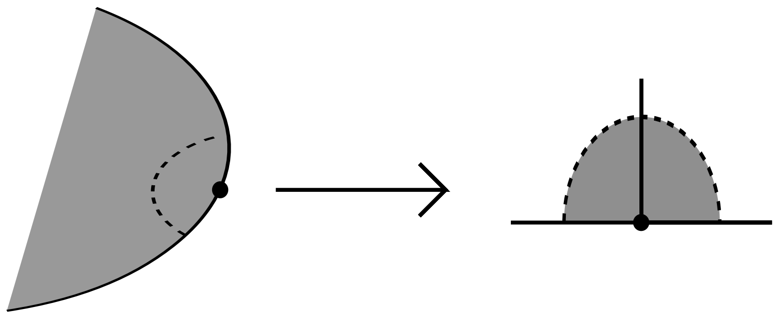
Definition 3.2. A **surface** is a collection of points $S \subset \mathbb{R}^n$ such that

- for any point x in S,
- there is a open ball B_x in $\mathbb{R}^n$ centered at x such that
- there is a function $f : S \cap B_x \to \{(x, y) : x^2 + y^2 < 1\}$ sending $x \to 0$ that's 1-1, onto, and continuous with continuous inverse.

Put in plain english, a **surface** is a collection of points that looks like the plane everywhere, if you look closely enough – i.e. there are small neighborhoods about every point that look like the open unit disk.

Examples of surfaces are spheres, tori, cubes, polyhedra, the plane $\mathbb{R}^2$ itself, and several other objects. Nonexamples of surfaces are solid n-dimensional shapes when $n \geq 3$ (i.e. a solid ball in $\mathbb{R}^3$ itself), $\mathbb{R}^3$ itself, a pair of spheres glued together at some single point (as no neighborhood of that point can look like an open disk), curves and other one-dimensional shapes, and many other objects.

However, this raises a question: what do we make of something like the disk $\{(x, y) : x^2 + y^2 \leq 1\}$? Your text and many theorems treat this as a surface, yet under the definition above it's not a surface – explicitly, there are no open neighborhoods of points around the boundary that look like an open disk! Rather, they look like open half-disks, as shown below:



So: this motivates the following definition of a surface with boundary, given below:

Definition 3.3. A **surface with boundary** is a collection of points $S \subset \mathbb{R}^n$ such that

- for any point x in S,
- there is a open ball B_x in $\mathbb{R}^n$ centered at x such that
- there is either a function $f : S \cap B_x \to \{(x, y) : x^2 + y^2 < 1\}$ sending $x \to 0$ that's 1-1, onto, and continuous with continuous inverse, **or**
- there is a function $f : S \cap B_x \to \{(x, y) : x^2 + y^2 < 1, x \geq 0\}$ sending $x \to 0$ that's 1-1, onto, and continuous with continuous inverse.

Basically, a surface with boundary is a collection of points such that if you zoom in enough, it locally looks either like the plane $\mathbb{R}^2$ or the upper-half-plane $\mathbb{H}^+ = \{(x, y) : y \geq 0$. We call the collection of all points around which S looks like $\mathbb{R}^2$ points in the **interior** of S, and the collection of all points where S looks like $\mathbb{H}^+$ the **boundary** of S. (The closed unit disk is a surface with boundary equal to the unit circle, for example.)

So: in your text, a surface is either a surface or a surface with boundary; in mathematics, people tend to use the word surface even when they're referring to a surface that may have a boundary.

4. Stokes's Theorem

So: with these definitions now firmly locked down, we move on to a rather powerful theorem:

Theorem 4.1. *(Stokes's Theorem) If F is a $\mathbb{C}^1$ vector field and S is a surface with boundary, we have that*

$$\int\int_S \nabla \times F \cdot dS = \int_{\partial S} F \cdot ds,$$

where by ∂S we denote the collection of curves that make up the boundary of S, each parametrized so that the surface S lies on the left-hand side of the curve when it is traversed (for more information, see last week's notes.)

So: this is another result in the vein of Green's theorem and the like. We illustrate its use below:

Example 4.2. For any surface without boundary (i.e. surfaces of the first kind that we defined above) and any $\mathbb{C}^1$ vector field F, we have that

$$\int\int_S \nabla \times F \cdot dS = 0$$

Proof. This follows trivially from Stokes's theorem, as such surfaces have no boundary, and thus

$$\int\int_S \nabla \times F \cdot dS = \int_{\partial S} F \cdot ds = \int_{\emptyset} F \cdot ds = 0.$$

□

Example 4.3. For $F = (2y, 0, 0)$ and S the upper half of the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$ find $\int\int_S F dS$.

Proof. So: note that we can write $F = \nabla \times (0, 0, -y^2)$. As a result, we have that

$$\int\int_S F dS = \int\int_S \nabla \times (0, 0, -y^2) ds;$$

applying Stokes to this integral gives us (as ∂S is traversed in the correct direction by $c(t) = (\cos(t), \sin(t), 0)$)

$$\int\int_S F dS = \int_0^{2\pi} (0, 0, -\sin^2(t)) \cdot (-\sin(t), \cos(t), 0) dt = 0.$$

□

5. Conservative Vector Fields

So, we change topics briefly to introduce a new object:

Definition 5.1. For F a C^1 vector field defined on all but a finite number of points in $\mathbb{R}^3$, we call F conservative if

$$\int_c F dS = 0,$$

for any simple closed curve c.

Theorem 5.2. *For F a C^1 vector field defined on all but a finite number of points in $\mathbb{R}^3$, the following conditions are equivalent:*

- *F is conservative; i.e.*

$$\int_c F dS = 0,$$

  for any simple closed curve c.
- *$F = \nabla f$, for some function f.*
- $\nabla \times F = 0$.

So: generally, when you want to show that something is conservative, try to show either the second or third property above (as making an argument for every possible curve will generally be hard); conversely, to show that something is not conservative, try to find a counterexample to the first property (i.e. find a curve on which its integral is nonzero), or again use the third property above if it's easy to take a derivative of. We work a pair of examples to illustrate these methods:

Example 5.3. We can see quickly that $F(x,y,z) = (x^2, y^2, z^2)$ is conservative, because it's the gradient of the function $f(x,y,z) = \frac{x^3+y^3+z^3}{3}$.

Example 5.4. We can see quickly as well that $F(x,y,z) = (0,x,0)$ is nonconservative, because its integral about the unit disk in the xy-plane

$$\int_{\partial D} F = \int_0^{2\pi} (0, \cos(t), 0) \cdot (-\sin(t), \cos(t), 0) dt = \frac{x + \cos(x)\sin(x)}{2}\Big|_0^{2\pi} = \pi \neq 0.$$

6. Gauss's Theorem

So: we have one more Green's theorem analogue (or, more properly, a divergence theorem analogue): Gauss's Divergence Theorem.

Theorem 6.1. *For W a closed volume in $\mathbb{R}^3$ with boundary equal to some surface ∂W, and F a smooth vector field, we have*

$$\int\int\int_W \nabla \cdot F dV = \int\int_{\partial W} F \cdot dS.$$

We work a pair of examples to show what's going on here:

Example 6.2. For S the unit sphere and F the vector field $(x, y, x^3 - 3xy^2)$, calculate $\int\int_S F dS$.

Proof. Because F is made of polynomials and the unit ball $B = \{(x,y,z) : x^2 + y^2 + z^2 \leq 1\}$ clearly has S as its boundary, we can apply Gauss's theorem to get

$$\int\int_S F dS = \int\int\int_B \nabla \cdot F dV = \int\int\int_B 2 dV = 2 \cdot vol(B) = 8\pi/3.$$

□

Example 6.3. For S the surface made of the faces of the unit cube $B = [0,1] \times [0,1] \times [0,1]$ and F the vector field (xy, yz, zx), calculate $\int\int_S F dS$.

Proof. Because F is again made of polynomials and the unit ball B again clearly has S as its boundary, we can apply Gauss's theorem to get

$$\int\int_S F dS = \int\int\int_B \nabla \cdot F dV = \int_0^1 \int_0^1 \int +0^1 x + y + z dV = 3/2.$$

□