MA1C, WEEK 1: LEVEL CURVES AND LIMITS

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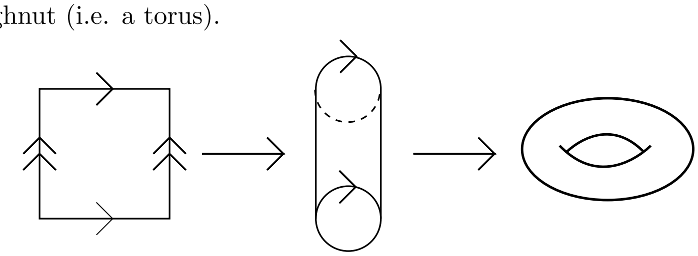
1. Administrivia and Announcements

So, before we begin, here's a series of random administrative things:

- These notes, like all future notes, can be found on my website.
- The easiest way to contact me if you have questions on the HW is via email! My address is padraic@caltech.edu.
- I have an office hour! From 8-9pm, on Sunday, in 155 Sloan (though if quite a few people show up, we'll just occupy a random nearby room.)
- The late HW policy has changed from last quarter. I.e. **you are no longer allowed to turn in late HW**. In the event that you become sick/calamities befall you/other such things, we need you to contact us by 10pm the night before the HW is due; furthermore, please only attempt this under fairly dire circumstances, as we have very little leeway to be merciful. If this is confusing, look at the course webpage, or contact any of us for further clarification! (as evidenced by the fact that I've messed the details of this policy up twice by attempting to go from memory, this is possible.)
- Draw all of your pictures for this HW by hand.

2. Random Questions

So: notice that if you glue together the sides of a square as depicted below, you get a doughnut (i.e. a torus).



Question 2.1. *Can you come up with a way to glue together a hexagon to get a torus? How about a way to glue together a octagon to get a 2-hole torus (i.e. a torus with two holes?) What other shapes can you make?*

3. Level Curves

So, the idea behind level curves is pretty simple: given a function $f(x, y)$, we can come up with a 3-dimensional graph for f by drawing the curves $C(a) = \{(x, y) : f(x, y) = a\}$, and putting those curves on the plane $z = a$ in the xyz-plane. If you've ever seen an elevation map or topographical map for a mountain range, this is exactly what we're doing, (except our functions here are not necessarily representing mountain ranges. but you get the idea.)

We work one explicit example here to give the idea of how these things go:

Example 3.1. Draw the level curves of the function

$$f(x, y) = \sqrt{16 - x^2 - y^2}$$

at the values $0, \sqrt{5}, \sqrt{12}, \sqrt{15}, 4$. What shape is this?

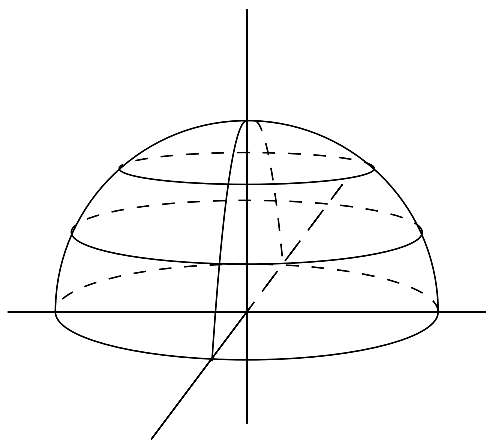
So: for $f(x, y) = 0$, this is going to just be the graph of the curve

$$0 = \sqrt{16 - x^2 - y^2};$$

i.e.

$$16 = x^2 + y2,$$

the circle of radius 4. Similarly, the level curves corresponding to $\sqrt{5}, \sqrt{12}, \sqrt{15}, 4$ will correspond to circles of radius 3, 2, and 0; graphing then gives us that the shape in question is



4. Limits - Definitions

So: back in first quarter, we had two equivalent definitions for what it meant for a single-variable function $f : \mathbb{R} \to \mathbb{R}$ to have a limit L at a point a. We review both of them below, briefly.

Definition 4.1. (epsilon-delta definition:) We say that

$$\lim_{x \to a} f(x) = L$$

if and only if

$$(\forall \epsilon > 0), (\exists \delta > 0) \text{ s.t. } (\forall x \text{ s.t. } |x - a| < \delta), (|f(x) - L| < \epsilon).$$

One way to understand this definition (kinda) is to imagine a two-player game, with play that goes through three rounds as follows:

- Player 1 names some constant, ϵ.
- Player 2, having heard player 1's constant, then responds with a second number δ.
- Player 1 then responds with any point x that's within distance δ of a.

We say that Player 2 wins if $f(x)$ is within distanceϵ from L, and Player 1 wins otherwise.

In this framework of a "game," we can then say that $\lim_{x\to a} f(x) = L$ holds if and only if Player 2 above has a strategy to win every time. (This concept of playing games as a proof strategy is something that comes up in mathematics, and has produced some remarkably intuitive proofs of complicated theorems.)

We also had a different (yet equivalent!) definition using the language of neighborhoods, which we describe below:

Definition 4.2. (neighborhood definition) We say that

$$\lim_{x\to a} f(x) = L$$

if and only if for every neighborhood U of L (remember: a neighborhood of a point is just an open set containing that point), there is a neighborhood V of a such that

$$f(V \setminus \{a\}) \subset U,$$

(where $V \setminus \{a\}$ denotes the set of all points in V that are **not** a.)

So: these were the one-dimensional definitions of limits. It turns out that the higher-dimensional definitions of limits are pretty much the same; i.e if we just replace all of the instances of one-dimensional variables with n-dimensional variables in the definitions above, we get the definitions of limits in the higher-dimensional sense! The reasoning for why this would be true is that limits are merely a way of formally talking about whether a function is "getting close" to something; and this is not a concept we would expect to be much different in the multidimensional sense.

We repeat the higher-dimensional definitions below, for your convenience. Here, we assume that f is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$, and consequently that $\bar{a}, \bar{x}$ are elements in $\mathbb{R}^n$ and $\bar{L}$ is an element in $\mathbb{R}^m$.

Definition 4.3. (epsilon-delta definition:) We say that

$$\lim_{\bar{x}\to \bar{a}} f(\bar{x}) = \bar{L}$$

if and only if

$$(\forall \epsilon > 0), (\exists \delta > 0) \text{ s.t. } (\forall \bar{x} \text{ s.t. } ||\bar{x} - \bar{a}|| < \delta), (||f(\bar{x}) - \bar{L}|| < \epsilon).$$

Definition 4.4. Alternately, we say that

$$\lim_{\bar{x}\to \bar{a}} f(\bar{x}) = \bar{L}$$

if and only if for every neighborhood U of $\bar{L}$, there is a neighborhood V of $\bar{a}$ such that

$$f(V \setminus \{\bar{a}\}) \subset U,$$

(where $V \setminus \{\bar{a}\}$ denotes the set of all points in V that are **not** $\bar{a}$.)

5. Limits - Worked Examples

So: this is well and good. But how do we use these definitions? We work several examples below:

Example 5.1. Does the function

$$f(x, y) = \frac{y^2}{x^2 + y^2}$$

have a limit at the point $(0, 0)$?

Proof. So, the answer to this question is no! To see why: consider the two lines $x = 0$ and $y = 0$ through the origin. Along the first line $x = 0$, this function $f(0, y)$ is identically equal to $\frac{y^2}{y^2} = 1$, so its limit as we approach 0 is 1; along the second line ($y = 0$), this function $f(x, 0)$ is identically 0 and so its limit as we approach 0 is 0.

So: we will do a proof by contradiction. Suppose this function f had a limit L. Then, we know that for any epsilon – say, $1/3$ – there must be some δ such that whenever $||(x, y)|| < \delta$, $|f(x, y) - L| < \epsilon$. But we know that $f(0, \delta/2) = 1$ and $f(\delta/2, 0) = 0$; so $|1 - L| < 1/3$ and $|L| < 1/3$. This is clearly impossible by the triangle inequality! So no limit exists.

Note that there was nothing special here about 1 and 0, except that they were not equal to each other. In general, to show that a limit of a does not exist, it suffices to find two distinct paths along which the function approaches different values, because we can always do an $\epsilon - \delta$ argument like the one above when that happens. □

Example 5.2.

$$f(x, y) = \frac{(2x - y)^{32}}{(2x)^{32} + y^{32}}$$

have a limit at the point $(0, 0)$?

Proof. So, this question is a touch more subtle. Along both of the paths $x = 0$ and $y = 0$, this function is identically 1, so we can't just use the paths we did before: however, if we examine the path $2x = y$, we have that $f(x, 2x) = 0$ for all nonzero x, so the function f goes to 0 along this path.

So, by the same argument as before, this function doesn't have a limit at 0. □

Example 5.3. Does

$$f(x, y) = \frac{\cos^2(x^2 y) - 1}{y^4 x^2}$$

have a limit at the point $(0, 0)$?

Proof. So: first, we simplify the function into the form

$$\begin{aligned} f(x, y) = \frac{\cos^2(x^2 y) - 1}{y^2 x} &= \frac{\sin^2(x^2 y)}{y^4 x^2} \\ &= \left(\frac{\sin(x^2 y)}{y^2 x}\right)^2. \end{aligned}$$

From here, we again consider two paths: the first of which is the line $y = x$. Along this path, we have that this function $f(x, x) = \left(\frac{\sin(x^3)}{x^3}\right)^2$ goes to 1, as

- x^3 goes to 0 as $x \to 0$,
- $\sin(x)/x$ goes to 1 as $x \to 0$,
- x^2 goes to 1 as $x \to 1$, and
- we can simply compose the limits of these functions.

However, along the path $x^2 = y$, we have that

$$f(x^2, x) = \left(\frac{\sin(x^4)}{x^5}\right)^2 = \frac{1}{x^2} \cdot \left(\frac{\sin(x^4)}{x^4}\right)^2;$$

As x goes to 0, we know by our earlier arguments that the $\left(\frac{\sin(x^4)}{x^4}\right)^2$-part of the equation above goes to 1. However, the $\frac{1}{x^2}$ part clearly goes to infinity as $x \to 0$; so f goes to infinity along this path. Since infinity and 1 are very different values, we can again conclude that no limit exists. □

However, occasionally, limits do exist! Here's an example of one:

Example 5.4. We claim that

$$\lim_{(x,y,z)\to 0} \cos(xyz) = 1.$$

Proof. To see why: notice that

$$|xyz| \leq \max(x^3, y^3, z^3) \leq ||(x, y, z)||^3$$

by the triangle inequality, and thus that $|xyz| \to 0$ as $(x, y, z) \to 0$. Because cosine is continuous, we can compose limits to get that

$$\lim_{(x,y,z)\to 0} \cos(xyz) = 1,$$

as desired. □

In general, you can do things like epsilon-delta proofs to show that limits exist; but often more elegant proofs can be devised by simply using continuity and results from the one-dimensional cases we already know.