Math 1b	TA: Padraic Bartlett

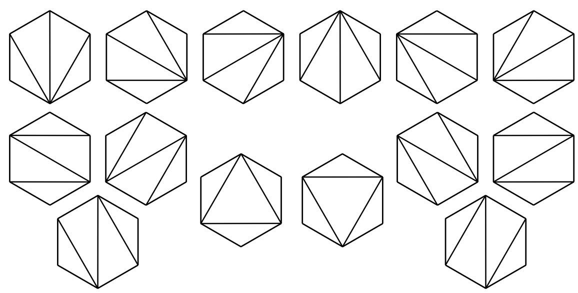
Recitation 6: Eigenthings

Week 6 *Caltech 2011*

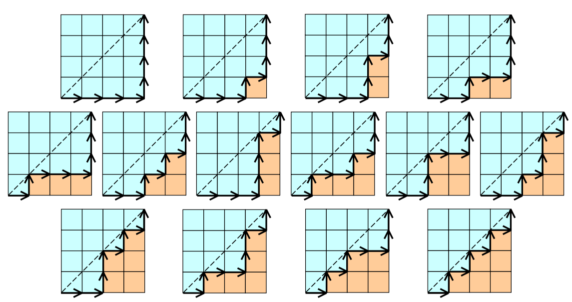
1 Random Question

Show that the following are equal[1]:

1. The number of ways to divide a regular $n+2$-gon into triangles, by connecting vertices with straight lines.



2. The number of ways to go from $(0, 0)$ to (n, n) on the integer lattice, via only walking north or east, so that you never go above the diagonal line $y = x$.



2 Midterm comments

- Section average: 70%.
- Basically, people semi-universally got questions 1/3/4 (the "calculational" questions) and had a lot of difficulty with 2/5 (the "proof" questions.) So, in terms of absorbing the Math 1b material, I'm not too concerned; most of the errors were either logical (not understanding "if and only if" statements; getting tripped up with how to even *parse* question 5) or just people being flustered.

[1]Pictures from Wikipedia's page on the Catalan numbers.

That the proofs are as rocky as they are, to be clear, *is* a concern; to fix this, we're going to try to do some more careful proofs in rec (whenever there's time?) and try to talk more about the "how" of proofs in office hours; after a week or two, if things haven't improved, we may try something more radical to prepare you all for the final. Honestly, I'm pretty happy with your collective performance thus far through the course! – I just want to make sure that we make it through the end intact. Relatedly: if you'd like different things in rec, tell me! I can adapt to student preferences.

- If you're concerned about your performance through the course thus far, please feel free to contact me! I can tell you how you're doing with respect to your peers, give you feedback as to what you can improve, set you up with tutoring or additional office hours, or just tell you you're doing absolutely fine (as the situation warrants.)

3 Eigenthings: Definitions

Definition. For a matrix A and vector $\mathbf{x}$, scalar λ, we say that λ is an **eigenvalue** for A and $\mathbf{x}$ is a **eigenvector** for A if and only if

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Equivalently, we can say the same things about λ and $\mathbf{x}$ if and only if

$$(\lambda \cdot I - A)\mathbf{x} = 0,$$

something that we can see by simply subtracting $A\mathbf{x}$ from both sides of our first equation. Based on this observation, we can see that λ is an eigenvalue for A if and only if $\det(\lambda I - A) = 0$.

Motivated by this, we define the **characteristic polynomial** of A as the polynomial

$$p_A(\lambda) = \det(\lambda I - A),$$

which we think of as a polynomial with variable given by λ.

Finally, for any eigenvalue λ, we can define the **eigenspace** E_λ associated to λ as the space

$$E_\lambda =: \{\mathbf{v} \in V : A\mathbf{v} = \lambda\mathbf{v}\}.$$

It bears noting that (using similar arguments to the ones above,) for any eigenvalue λ, we can also define

$$E_\lambda =: \text{nullspace}(\lambda I - A).$$

4 Eigenthings: Two Examples

We calculate two examples here, one along the computational/trivial end of the spectrum, and one more along the more conceptual/proof-based end of things:

Example. For $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, find all of A's eigenvectors, eigenvalues, and associated eigenspaces.

Solution. We start by finding A's eigenvalues; from there, we can find the eigenspaces attached to each eigenvalue, and thereby have found all of A's eigenvectors.

To do this, we simply examine

$$\begin{aligned}
\det(\lambda I - A) &= \det\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\right) \\
&= \det\left(\begin{pmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 2 \end{pmatrix}\right) \\
&= (\lambda - 1)(\lambda - 2).
\end{aligned}$$

This polynomial has $\lambda = 1, 2$ as its two roots; thus, 1 and 2 are our two eigenvalues.

We now proceed to find E_1 and E_2:

$$\begin{aligned}
E_1 &= \text{nullspace}(1 \cdot I - A) \\
&= \text{nullspace}\left(\begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}\right).
\end{aligned}$$

What vectors are such that $\begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} \cdot (x, y)^T = 0$? Well, ones such that

$$\begin{aligned}
0 \cdot x + (-1) \cdot y = 0, \text{ and} \\
0 \cdot x + (-1) \cdot y = 0;
\end{aligned}$$

i.e. vectors of the form $(x, 0)$. So $E_1 = \{(x, 0) : x \in \mathbb{R}\}$.

Similarly, we can find E_2 in the same way, by first noting that

$$\begin{aligned}
E_2 &= \text{nullspace}(2 \cdot I - A) \\
&= \text{nullspace}\left(\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\right),
\end{aligned}$$

which corresponds to the set of vectors $(x, y)^T$ such that

$$\begin{aligned}
1 \cdot x + (-1) \cdot y = 0, \text{ and} \\
0 \cdot x + 0 \cdot y = 0.
\end{aligned}$$

This forces our vectors to be of the form (x, x), and thus implies that $E_2 = \{(x, x) : x \in \mathbb{R}\}$.

This characterizes all of A's eigenvectors, values, and spaces.

Example. For an odd positive integer n, show that the only eigenvectors of the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

are of the form $(x, x \dots x)$,, where x is any real number.

Proof. So, this example seems a lot trickier than the one we just did – it's some sort of crazy $n \times n$ matrix, instead of a nice manageable 2×2 matrix! Yet, as it turns out, we can still solve this problem by just using the same methods that we used earlier, with just a bit more patience and cunning.

Specifically; to find the eigenvectors of A, we first want to find all of the possible eigenvalues of A: i.e. all of the values of λ such that $\det(\lambda I - A)$ is 0.

We start by first examining the matrix $(\lambda I - A)$ itself:

$$\lambda I - A = \begin{pmatrix} \lambda & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \lambda & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \lambda & -1 \\ -1 & 0 & 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$$

Expanding the determinant of this looks messy; while there are only two nonzero terms on the top row, the resulting matrix gets kind of weird when you delete only the second column and first row. So, to calculate the determinant of $\lambda I - A$, we will instead calculate the determinant of its *transpose*.

Why would we want to do that? Well, as it turns out, the determinant of the transpose is a lot easier to calculate, as expanding the determinant along its first row results in the sum of two determiants of triangular matrices, and we know that the determinant of a triangular matrix is just the product of its diagonal entries!

Explicitly, we have

$$\det(\lambda I - A) = \det((\lambda I - A)^T)$$
$$= \det \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ -1 & \lambda & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & \lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & \lambda & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & \lambda \end{pmatrix}$$
$$= \lambda \cdot \det \begin{pmatrix} \lambda & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & \lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & \lambda & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & \lambda \end{pmatrix} - 0 + \dots + 0$$
$$+ (-1)^{n-1} \cdot (-1) \cdot \det \begin{pmatrix} -1 & \lambda & 0 & 0 & 0 & \dots & 0 \\ 0 & -1 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & \lambda & 0 & \dots & 0 \\ 0 & 0 & 0 & -1 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 \end{pmatrix},$$

where the $(-1)^{n-1}$ above comes from the definition of the determinant (i.e. because $\det(A) = \sum_{i=1}^{n}(-1)^i a_{1i} \det(A_{1i})$.) So, again, if we recall that the determinant of a triangular matrix is just the product of the entries along the diagonal[2], we have that

$$\begin{aligned} \det(\lambda I - A) &= \lambda \cdot \lambda^{n-1} + (-1)^{n-1} \cdot (-1) \cdot (-1)^{n-1} \\ &= \lambda^n + (-1)^{2n-1} \\ &= \lambda^n - 1. \end{aligned}$$

If n is odd, this only has one real-valued solution: $\lambda = 1$. If n is even, this has exactly two real-valued solutions: $\lambda = 1$ and $\lambda = -1$.

So; in the case that n is odd, let's classify E_1, which (as 1 is the only eigenvalue) will tell us what all of A's eigenvectors are.

[2]if you don't remember this proof from the notes/class, it's easy to do by induction! Specifically, it's trivially true for a 1×1 matrix: now, start with a lower-triangular matrix, and look at its first row: one nonzero entry and the rest are zeroes. expand along that row, and apply your inductive hypothesis! For upper-triangular: use the fact that $\det(A^T) = \det(A)$,, and the result you just proved for lower-triangular matrices!

By definition,

$$E_1 = \text{nullspace}\,(I - A)$$
$$= \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

We *can* use our established methods for finding general nullspaces to find the nullspace of this matrix; but it's kind of obnoxious and not as clear. Instead, to find the nullspace of this matrix, we can make the following two observations:

- If you add up the first $n-1$ rows of our matrix, you get the last row of our matrix. Therefore, the nullspace of $I - A$ is at least 1-dimensional.
- If you take any nontrivial[3] linear combination of the first $n-1$ rows of our matrix, you do not get zero, as the 1-part of the earliest row we pick can't be cancelled out by any of the other rows. (Think about this for a second if you don't believe it.)

  Therefore, the nullspace of our matrix is at most $n - (n-1) = 1$-dimensional.
- If we multiply the vector $(x, x, \dots x)^T$ on the right by A, we just get $(x, x, \dots x)$ back; therefore, this vector is in the nullspace of $I - A$.

So: we've shown that the nullspace of $I - A$ is one-dimensional and contains the vector $(x, x \dots, x)$,for any real value x: therefore, it *is* made of those vectors. In other words, we've shown that

$$E_1 = \{(x, x, \dots, x) : x \in \mathbb{R}\};$$

thus, every eigenvector of A is of this form.

For fun: what happens when n is even? Do you get any new eigenvectors?

[3] A linear combination $\sum a_i x_i$ of some collection of x_i's is called nontrivial if at least one of the coefficients a_i is nonzero.