Math 1b	TA: Padraic Bartlett

Recitation 10: Reflections, Isometries, SVD's, and Pseudoinverses

Week 10 — *Caltech 2011*

1 Random Question

Can you find a 20070201×20070201 matrix with all of its entries either equal to 0 or 1, such that its determinant is 2?

2 Homework comments

- Average: roughly 86.5%, which was a few points above the class average. Good job!
- Comments: The only thing I want to say is that when you're diagonalizing a symmetric matrix A with an orthogonal matrix E, **make sure to normalize the columns of** E! Otherwise, you don't have an orthogonal matrix; this made it so that when people calculated $E^T AE$, they didn't get D as expected.

So: as you may have noticed from lecture / the notes online: class went insane in the last week! Specifically, we've covered the concepts of isometry, reflections, singular value decompositions, pseudoinverses, and all of their applications in like three lectures, which is kind-of crazy. But not too crazy for you all to not own the concepts on the final! In this recitation, we're going to review the basics about reflections and isometries; this, coupled with the final review's notes on SVD's and pseudoinverses, should set you up nicely. (For those of you who were at this recitation; I've moved the SVD discussion we did in class to the final review notes, because otherwise I'd just be repeating the same thing twice.)

3 Isometries and Reflections: Definitions

So: what *is* an isometry, anyways?

Definition. Isometry: A **isometry** is a map $f : \mathbb{R}^n \to \mathbb{R}^n$ that preserves distances: i.e. for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,, we have

$$||\mathbf{x} - \mathbf{y}|| = ||f(\mathbf{x}) - f(\mathbf{y})||$$

Intutively, we think of isometries of ways of moving space around that doesn't "bend" it in any key ways – they're maps that preserve distances, and thus don't accidentally fold two points closer to each other.

We've worked with several examples of isometries – rotations, for one, are a common choice of isometry. Another map that is an isometry is the **reflection map**, which we've defined before but will restate here:

Definition. Reflection: For a subspace U of $\mathbb{R}^n$, we define the **reflection map through U** as the linear map

$$\mathrm{Refl}_U(\mathbf{x}) = \mathbf{x} - 2 \cdot \mathrm{proj}_{U^\perp}(\mathbf{x}).$$

One question we can ask here (that we ask about pretty much everything we study in this course) is the following: how can we describe this process via **matrices**? In other words:

Question 1 *For any subspace $U \subseteq \mathbb{R}^n$, what is the matrix R_U such that for any vector $\mathbf{x} \in \mathbb{R}^n$, $R_U \cdot \mathbf{x} = \mathit{Refl}_U(\mathbf{x})$?*

Solution. As it turns out, yes! We outline the construction of such "reflection matrices" below.

Suppose we have some subspace U of $\mathbb{R}^n$, of dimension $k \leq n$. Choose a basis $\{\mathbf{u}_1, \ldots \mathbf{u}_k\}$ s for U. As well, notice that the subspace $U^\perp$ has dimension $n-k$, and choose a basis $\{\mathbf{w}_1, \ldots \mathbf{w}_{n-k}\}$ for $U^\perp$.

Now, look at what $\mathrm{Refl}_U(\mathbf{x})$ does to each of these basis elements. We know that $\mathrm{Refl}_U(\mathbf{x})$ fixes each of the $\mathbf{u}_i$'s, because reflecting through the subspace U fixes all of the elements in U. As well, we know that $\mathrm{Refl}_U(\mathbf{x})$ sends each of the $\mathbf{w}_i$'s to $-\mathbf{w}_i$'s, because reflecting through U sends elements in $U^\perp$ to their additive inverses, by definition.

But what does this mean? Well, that U is a map with

- 1 as an eigenvalue of geometric multiplicity k, with eigenvectors$\{\mathbf{u}_1, \ldots \mathbf{u}_k\}$, and
- -1 as an eigenvalue of geometric multiplicity $n-k$, with eigenvectors$\{\mathbf{w}_1, \ldots \mathbf{w}_{n-k}\}$.

But this means we've found n linearly independent eigenvectors for U! So, we can reverse-engineer our diagonalization process we discovered earlier, to notice that the matrix R_U given by the product

$$\begin{pmatrix} | & & | & | & & | \\ \mathbf{u}_1 & \ldots & \mathbf{u}_k & \mathbf{w}_1 & \ldots & \mathbf{w}_{n-k} \\ | & & | & | & & | \end{pmatrix} \cdot \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix} \cdot \begin{pmatrix} | & & | & | & & | \\ \mathbf{u}_1 & \ldots & \mathbf{u}_k & \mathbf{w}_1 & \ldots & \mathbf{w}_{n-k} \\ | & & | & | & & | \end{pmatrix}^{-1}$$

has precisely the same eigenvalues and eigenvectors as the Refl_U map. Therefore, because we can write any vector in $\mathbb{R}^n$ as a sum of these eigenvectors (because we found n of these eigenvectors and they're all linearly independent $\Rightarrow$ they form a basis for $\mathbb{R}^n$), we know that because Refl_U and R_U have the same eigenvectors/eigenvalues, they must send the same vectors to the same places. Therefore, they are the same map!

Therefore, we can express any reflection map as the product R_U of matrices above.

The details of the proof above aren't too important; more useful is the result of the theorem above, which gives us an explicit way to write down any reflection as a matrix. We give an example of this below:

Question 2 *Find the 2×2 matrices for reflection through the line $x = y$, and for reflection through the line $y = 0$. What is the product of these two matrices?*

Solution. We use our results and methods above. Specifically, let's start with the matrix that's reflection through the line $x = y$. To phrase the question differently, this is reflection through the subspace $U = \{(x, y) : x = y\}$, the orthogonal complement of which is the subspace $U^\perp = \{(x, y) : x = -y\}$. (To see this, you can either use the definition that the orthogonal complement $U^\perp$ consists of all vectors that are orthogonal to U, or the geometric definition that in $\mathbb{R}^2$, the orthogonal complement to a line through $(0, 0)$ is the line through $(0, 0)$ that's perpindicular to it.)

As U is one-dimensional, a basis for it is just any element in U; for example $\{(1, 1\}$. Similarly, as $U^\perp$ is one-dimensional, we can give its basis as $\{(1, -1)\}$ without any work. Then, by our work earlier, we can write

$$\begin{aligned} R_U^1 &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Similarly, to find the matrix for reflection through the line $x = 0$, we can write $U = \{(x, y) : x = 0\}$, $U^\perp = \{(x, y), y = 0\}$, pick $\{(0, 1)\}$ to be a basis for U and $\{(1, 0)\}$ to be a basis for $U^\perp$, and write

$$\begin{aligned} R_U^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

So, if we take the product of these matrices, we get

$$\begin{aligned} R_U^2 \cdot R_U^1 &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

which (on inspection) is the matrix corresponding to counterclockwise rotation by $\pi/2$ around the origin.

If you check this geometrically, this should make sense; on paper, flipping through the line $x = y$ and then through the line $x = 0$ should yield the $\pi/2$ rotation we got. Check it for yourself!