CRAMER'S RULE AND EIGEN(VECTORS/SPACES/VALUES)

PADRAIC BARTLETT

1. Random Questions

Question 1.1. *What are all of the eigenvectors and eigenvalues for the matrix*

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \ddots & \\ 0 & \dots & & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}?$$

Question 1.2. *Is there a 20050705 by 20050705 real-valued matrix such that*

- *every entry is either 0 or 1, and*
- *its determinant is 2?*

2. Cramer's Rule

Theorem 2.1. *For a $n \times n$ matrix*

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \\ a_{n,1} & a_{n,2} & \dots & a_{1,n} \end{pmatrix}$$

such that $\det(A) \neq 0$, and a vector b, we have that there is a solution $Ax = b$ with the vector $x = (x_1 \dots x_n)$ explicitly given by the formulas

$$x_i = \frac{\det \begin{pmatrix} a_{1,1} & \dots & a_{1,i-1} & b_1 & a_{1,i+1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,i-1} & b_2 & a_{2,i+1} & \dots & a_{2,n} \\ \vdots & & & & & \ddots & \\ a_{n,1} & \dots & a_{n,i-1} & b_n & a_{n,i+1} & \dots & a_{n,n} \end{pmatrix}}{\det(A)}$$

So: this is a pretty useful computational tool for solving simple linear systems by hand quickly. We work an example below:

Example 2.2. Use Cramer's rule to solve the linear system of equations $Ax = b$, for

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 2 & 0 \\ 6 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

So: because

$$\det A = 0 \cdot (2-0) - 1(3-0) + 0(0-12) = -3 \neq 0,$$

we can in fact apply Cramer's rule – if this didn't hold, we would be completely unable to say anything about what the solutions to this linear system would be (or even if they existed!), because Cramer's rule involves division by the determinant to define the x such that $Ax = b$.

So: let's calculate x! (This is being excited, not a factorial.)

$$x_1 = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 0 & 1 \end{vmatrix}}{\det(A)} = \frac{1(2-0) - 1(2-0) + 0(0-6)}{-3} = 0,$$

$$x_2 = \frac{\begin{vmatrix} 0 & 1 & 0 \\ 3 & 2 & 0 \\ 6 & 3 & 1 \end{vmatrix}}{\det(A)} = \frac{0(2-0) - 1(3-0) + 0(3-0)}{-3} = 1,$$

$$x_3 = \frac{\begin{vmatrix} 0 & 1 & 1 \\ 3 & 2 & 2 \\ 6 & 0 & 3 \end{vmatrix}}{\det(A)} = \frac{0(6-0) - 1(9-12) + 1(0-12)}{-3} = 3$$

$$\Rightarrow x = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}.$$

As a reality check to make sure that we calculated everything, we can check quickly that

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 2 & 0 \\ 6 & 0 & 1 \end{pmatrix} \cdot x = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 3 \\ 3 \cdot 0 + 2 \cdot 1 + 0 \cdot 3 \\ 6 \cdot 0 + 0 \cdot 1 + 1 \cdot 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = b,$$

as predicted by the theorem.

3. Eigen(vectors/spaces/values)

Definition 3.1. For a matrix A, we call λ an **eigenvalue** of A if λ is not equal to 0 and there is some vector x such that

$$A \cdot x = \lambda x.$$

We call such vectors x **eigenvectors** for A; in a sense, the eigenvectors of A can be thought of as the vectors which A sends back to their "own" spaces (hence the use of the german word "eigen," which translates to "own.")

Equivalently, we can call λ an eigenvalue of A if λ is such that the determinant of the matrix

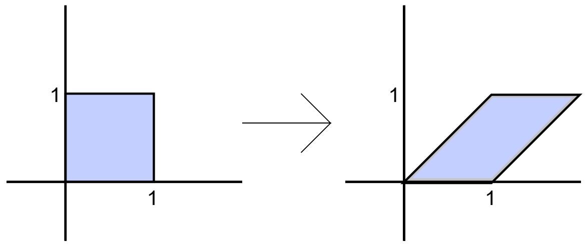
$$\lambda I - A$$

is 0; in this case, we think of an eigenvector corresponding to λ as an element in the null space of $\lambda I - A$.

For a given eigenvalue λ, we can denote the space spanned by all λ-eigenvectors as the **eigenspace** corresponding to λ; this space is equal to the null space of $\lambda I - A$.

So, this is all well and good – but what (you ask) are some examples of eigenvalues and spaces?

Well: consider the following map:



Visually, it's immediate that this map has one eigenvalue, 1, corresponding to the vector $(1, 0)$; but how can we see this with our complicated definitions?

So: this map sends $(0, 1)$ to $(1, 1)$, and $(1, 0)$ to $(1, 0)$, and thus corresponds to the matrix

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};$$

this matrix has 1 as its only eigenvalue, as $\det(\lambda I - M) = (1 - \lambda)^2 = 0$ iff $\lambda = 1$.

As a result: all eigenvectors of this map must lie in the null space of the matrix

$$1 \cdot I - M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

this matrix has rank 1, and thus its null space has rank 1 and is spanned by any nontrivial element in it – in particular, it's spanned by $(1, 0)$,, as $(I - M)(1, 0) = (0, 0)$.

So we've completely categorized all of the eigenvalues, vectors, and spaces of this linear map! We can use these methods to successfully categorize most maps that will be dealt with in this class.