BASIS, DIMENSION AND ORTHOGONALITY

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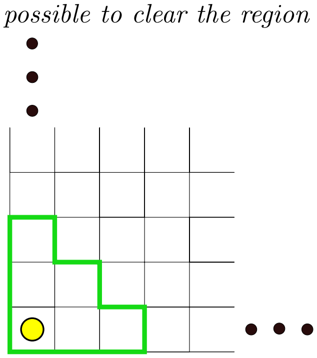
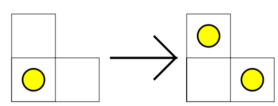
1. Random Questions

Question 1.1. *So: Suppose you have a* $\mathbb{N} \times \mathbb{N}$ *grid of squares – kind of an infinte-dimensional chessboard with a lower-right-hand corner, if you will – and consider the following game we can play on our board:*

- *We start by putting one coin on the square in the bottom-right-hand corner.*
- *If we have a coin on some square on our board such that the square immediately north of the coin and the square immediately east of the coin are both empty, we can remove the coin from our square and put a new coin to the north and a new coin to the east of this square.*

So: Is it possible to clear the region highlighted in green below?





2. Last Week's Homework

Average: about 63. Interesting points about the HW: not really much. As evidenced by the average, people seemed to know what was going on. However, if anything was shaky, or you couldn't read the comments I wrote next to a given problem, feel free (as always!) to contact me and I'll try to clarify things.

3. Basis!

Definition 3.1. So: we define a nonempty set U of vectors to be a **space** if it is closed under linear combinations (i.e. scalar multiples of elements of U and sums of elements of U are again elements of U).

Definition 3.2. For a space U, a **basis** of U is a linearly independent set of vectors $\{v_i\}$ that span U – i.e. $\langle v_i \rangle = U$.

Examples 3.3. $\mathbb{R}^n$ has a basis (the standard basis, natch) given by the vectors $e_i = (0, \ldots, 1, \ldots, 0)$, where the 1 occurs in the i-th place and all other entries are 0.

The space of polynomials $R[x]$, where R can be taken to be $\mathbb{Z}, \mathbb{R}, \mathbb{Q}$ or a host of other things (usually, these are at least rings)(if you don't know what a ring is, ignore that comment), has a basis consisting of the polynomials

$$1, x, x^2, x^3, x^4 \ldots$$

So: given a space, how do we find a basis? The process is very algorithmic (i.e. there's a simple step-by-step approach that constructs it for you) – the proof below illustrates how it goes in general.

Theorem 3.4. *Any space $U \subset \mathbb{R}^n$ has a basis.*

Proof. (This holds for all spaces; but in infinite-dimensional cases it gets tricky, and it's more illuminating to do the case that you're going to work in all the time.)

So: U is a space, and thus nonempty. If U consists solely of the 0-vector, then U is trivially spanned by the empty set; otherwise, there is a $v_1 \in U$ such that $v_1 \neq 0$.

Consider the set $U \setminus \langle v_1 \rangle$ (where for two sets A, B, $A \setminus B$ is defined to be the collection of all elements in A that are not elements in B).

Two cases can hold: either

(1) $U \setminus \langle v_1 \rangle$ is empty. In this case, halt.
(2) $U \setminus \langle v_1 \rangle$ is not empty. In this case, pick some element v_2 in U that is not 0 and not in $\langle v_1 \rangle$.

From here, again, two cases can hold: either

(1) $U \setminus \langle v_1, v_2 \rangle$ is empty. In this case, halt.
(2) $U \setminus \langle v_1, v_2 \rangle$ is not empty. In this case, pick some element v_3 in U that is not 0 and not in $\langle v_1, v_2 \rangle$.

Repeat this process until at some v_k we have that $U \setminus \langle v_1 \ldots v_k \rangle$ is empty – such a stage exists because the space U is contained in $\mathbb{R}^n$, and thus cannot contain more than n-linearly independent vectors. (question: why is this true?)

So: this collection trivially spans U, as every element in U lies in their span and they are all elements of U. That they are linearly independent is not much harder to see: pick any c_i such that

$$\sum_{i=1}^{k} c_i v_i = 0.$$

Because we picked v_k such that it didn't lie in the span of $\langle v_1 \ldots v_{k-1} \rangle$, we know that

$$\sum_{i=1}^{k} c_i v_i = 0 \Leftrightarrow \sum_{i=1}^{k-1} c_i v_i = -c_k v_k \Leftrightarrow c_k = 0;$$

so we have reduced our collection to

$$\sum_{i=1}^{k-1} c_i v_i = 0.$$

Repeating this process k times gives us that all of the c_i are necessarily 0; so the v_i are linearly independent, and thus are a spanning set! □

Example 3.5. The above process works to find basises in concrete settings. Take, for example,

$$U = \langle (1, 2, 3), (4, 5, 6), (7, 8, 9) \rangle.$$

U is not empty, so take $v_1 = (1,2,3)$. $U \setminus \langle (1,2,3) \rangle$ is also not empty, because the vector $(7,8,9)$ isn't a scalar multiple of $(1,2,3)$; so set $v_2 = (7,8,9)$.

Then, because

$$(4,5,6) = \frac{1}{2} \cdot ((7,8,9) - (1,2,3)) + (1,2,3),$$

we have that $U \setminus \langle (1,2,3), (7,8,9) \rangle$ is empty; thus, by the reasoning we used in the proof above, $(1,2,3), (7,8,9)$ form a basis for U.

Definition 3.6. A space U has **dimension** n if it has a basis with n elements. Theorems we will one day prove will show that this is indeed a well-defined concept (i.e. a space can't have basises of two different sizes), and that this concept accords with our intuitive notion of dimension (i.e. dimension 1 is a line, 2 is a plane, 3 is 3-d space, ...)

4. Orthogonality

Definition 4.1. v_1 is **orthogonal** to v_2 iff $v_1 \cdot v_2 = 0$. We denote this relation by writing $v_1 \perp v_2$. For spaces U, we define $U^\perp$ as the collection of all vectors v such that for all $u \in U$, $v \perp u$.

Geometrically, the notion of orthogonality coincides with the concept of perpindicularity – i.e. it turns out that

- $v_1 \cdot v_2 = |v_1||v_2| \cos(\theta)$, for θ the angle between the two vectors v_1 and v_2, and consequently that
- $v_1 \perp v_2$ iff the angle between them is $\pi/2$, i.e. 90 degrees.

You won't need this for the HW, but I thought it was cool to know, and maybe will help with your intuition for what's going on here, and why we would even care about orthogonality.

Protip: just as we showed that every space has a basis, it turns out that they can be made to have an orthogonal basis! (i.e. a basis where any two basis vectors are orthogonal to each other.) This is a cool fact, and we'll hopefully do that sometime this quarter – the process is called Gram-Schmidt orthogonalization, and is rather useful in a lot of things.