LINEAR SPANS, AFFINE SPANS, AND CONVEX HULLS

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1. Last Week's Homework

Average for last week: 85.

2. Linear Independence, Dependence, and Spans

Definition 2.1. A collection of vectors $\{v_i\}_{i \in I}$ is called **linearly dependent** if there exists a collection $v_{n_1}, \ldots v_{n_k}$ and scalars a_i, not all zero, such that

$$\sum_{1}^{k} a_i \cdot v_{n_i} = 0;$$

it is called **linearly independent** if no such combinations exist.

Furthermore, given a collection of vectors $A = \{v_i\}_{i \in I}$, we can define the **linear span** of A, $span(A)$, to be the collection of all finite linear combinations

$$\sum_{1}^{k} a_i \cdot v_{n_i},$$

where the a_i are all scalars and the v_{n_i} are elements of A. Intuitively, we think of $span(A)$ as the collection of all vectors that are linearly dependent on elements of the set A.

Remark 2.2. In the case where the vectors $\{v_i\}_{i \in I}$ are elements of $\mathbb{R}^n$, we have a "fairly intuitive" geometric interpretation of what their spans are, and what linear independence is.

Specifically: take a pair of vectors v_1, v_2 in $\mathbb{R}^n$ that are linearly independent. Then the span of v_1 and v_2 can be thought of as all possible combinations $xv_1 + yv_2$, for $x, y \in \mathbb{R}$. Because v_1 and v_2 are linearly independent, there is no pair of real numbers x, y such that $xv_1 + yv_2 = 0$ and either x or y is nonzero; so we can actually "think" of $span(v_1, v_2)$ as being the collection of all pairs (x, y), as each pair corresponds in a bijective fashion to an element of $span(v_1, v_2)$.

This identification is a way to see that two linearly independent vectors "span" a plane; similarly, we have that three linearly independent vectors will span a "space" (i.e. something that looks like $\mathbb{R}^3$, and n linearly independent vectors will span a n-space.

Remark 2.3. Recall this question, from week 1:

Question 2.4. *(Challenge) What is the largest set of vectors in $\mathbb{R}^3$ such that any three are linearly independent? (We say a collection of n vectors in $\mathbb{R}^3$ is **linearly independent** if the rank of the $n \times 3$ matrix formed by those n vectors is n.)*

Given our current geometric interpretation, we can rephrase the question into

Question 2.5. *What is the largest set of vectors in* $\mathbb{R}^3$ *such that any three do not lie in the same plane?*

So: consider the collection S of all vectors v such that the z-coordinate of v is $1/2$ and $|v| = 1$. No three points in S lie in the same plane through the origin, as a plane through the origin intersects the circle S in at most two points. This is thus an infinite set that satisfies our hypothesises above! So the answer is infinite – i.e. that there is no "largest" set.

3. Affine Independence, Dependence, and Spans

Definition 3.1. A collection of vectors $\{v_i\}_{i\in I}$ is called **affinely dependent** if there exists a collection $v_{n_1}, \dots v_{n_k}$ and scalars a_i such that $\sum a_i = 0$ and the a_i ar enot all 0, such that

$$\sum_1^k a_i \cdot v_{n_i} = 0;$$

it is called **affinely independent** if no such combinations exist.

Furthermore, given a collection of vectors $A = \{v_i\}_{i\in I}$, we can define the **affine span** of A, $span_a(A)$, to be the collection of all finite linear combinations

$$\sum_1^k a_i \cdot v_{n_i},$$

where the a_i are all scalars such that $\sum a_i = 1$ (notice the 1 here!) and the v_{n_i} are elements of A. Intuitively, we think of $span_a(A)$ as the collection of all vectors that are affinely dependent on elements of the set A; this is because for any element

$$v = \sum_1^k a_i \cdot v_{n_i},$$

we can write

$$0 = 1 \cdot v - \sum_1^k a_i \cdot v_{n_i},$$

where $1 - a_1 - \ldots - a_k = 0$, and not all of the entries are 0.

Remark 3.2. These definitions are fairly similar to those of linear dependence and independence – it is clear from the definitions above that an affinely dependent set is a linearly dependent set, and that a linearly independent set is an affinely independent set.

In the case where the vectors $\{v_i\}_{i\in I}$ are elements of $\mathbb{R}^n$, we again have a geometric interpretation of what their affine span is: specifically, as before, take a pair of vectors v_1, v_2 in $\mathbb{R}^n$ that are affinely independent.

Then the affine span of v_1 and v_2 can be thought of as all possible combinations $xv_1 + yv_2$, for $x, y \in \mathbb{R}$ such that $x + y = 1$. What does this set look like?

(insert picture here)

As we can see in the picture above, it looks like a line – this is because it is the one-dimensional set $\{xv_1 + (1 - x)v_2\}$ (one-dimensional here because there is one free variable; also because a line is 1-d.) Extending to three affinely independent points gives you the picture below,

(insert picture here)

which demonstrates that the affine span of three affinely independent points is a plane. Extending to 4 points gives a space: in general, n points will affinely span a $n-1$ dimensional space.

4. Convex Hulls

Finally, we have just one more concept:

Definition 4.1. We call a set X **convex** if for any two points $a, b \in X$ and $0 \leq \lambda \leq 1$, $\lambda a + (1-\lambda)b \in X$.

Furthermore, given a collection of points $A = \{x_i\}_{i \in I}$, we can define their **convex hull**, *hull*(A), to be the collection of all finite linear combinations

$$\sum_{1}^{k} a_i \cdot v_{n_i},$$

where the a_i are all scalars such that $\sum a_i = 1$ and all of the a_i are positive.

Remark 4.2. One last bit of intuition: what, geometrically speaking, does the convex hull of a set of points look like? Well – it looks like the affine hull of that set of points, except we are restricting the combinations we allow to ones where the a_i are positive. I.e. the convex hull of two distinct points x, y forms a line *segment*, as seen below;

(insert picture here)

the convex hull of three noncollinear points forms a triangle,

(insert picture here)

and the convex hull of four noncollinear points forms either a quadrilateral, a triangle, or a tetrahedron.

(insert picture here)