FINAL REVIEW NOTES

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1. Previous HW

Average was around 90% or so; there were only two things I wanted to stress for people going into the final.

(1) Always explain *why* something works! When concluding something, don't just write down that "the matrix is (blah)" – explain *why* it is (blah). The methods you use to arrive at your conclusions are far more important than your ability to multiply two-digit numbers successfully – yet, if you only write down a simple numeric answer and somewhere along the line a calculation went wrong, I as a grader cannot tell whether your problem came from something trivial (and should be penalized lightly) or from something grievously wrong. Words are your friends!

(2) When answering questions about a probability matrix P with associated adjacency graph G, several of you invoked relations between the concepts of G being **strongly connected**, $P^m > 0$ for some m, and **P**'s stable vector being > 0 that weren't always quite right. So I just want to clarify that here:

$$(\exists m\ P^m > 0) \Leftrightarrow (P's \text{ stable vector being} > 0) \Rightarrow (\text{G being strongly connected})$$

Proof. To see this: note that if P^m is bigger than 0 for some m, then (from the notes/recitation) we know that there are paths of length m between any two nodes in G, and thus that G is strongly connected. As well, if P^m is composed of strictly positive entries, by stringing together the paths of length m we got above we have paths of length km for any $k \in \mathbb{N}$ between any two nodes in G, and thus that P^{km} is > 0 for all k. Then, because P^m converges to the matrix composed of n copies of P's stability vector as m goes to infinity, we have that P's stability vector must be > 0, as P^m is infintely often composed of strictly positive entries. Finally, note that if P's stability vector is composed of strictly positive numbers, we would have that eventually P^m must be > 0, as it approaches the matrix composed of n copies of the stability vector. $\square$

The important thing to take away from this is that

$$(G \text{ being strongly connected}) \not\Rightarrow (\exists m\ P^m > 0):$$

to see an explicit example, take $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: P has a strongly connected adjacency matrix, yet P^m is always equal to either P or I, for any m.

2. Review topics!

So: here is a list of things (I think exhaustive, but don't hold me to it) that I think you'll need for the midterm.

- **Characteristic polynomial**: For a given matrix M, the polynomial of one variable x given by $\det(M - xI)$.
- **Eigenspaces**:For a matrix M and an eigenvalue $\lambda \in \mathbb{C}$, the eigenspace corresponding to λ is the nullspace of $(M - \lambda I)$: equivalently, it can be defined as the collection of all vectors v such that $Mv = \lambda v$.
- **Eigenvalues**: For a matrix M, eigenvalues are numbers $\lambda \in \mathbb{C}$ such that $\det(M - \lambda I) = 0$; equivalently, numbers λ such that there exist nonzero vectors v such that $Mv = \lambda v$.
- **Eigenvectors**: For a given eigenvalue λ, an eigenvector is a vector in λ's associated eigenspace.
- **Linear Programming**: Know how to do it. Prof. Wilson's notes online are (I believe) rather clear and concise on the subject, so I'll let them stand as a reference – but if you have any questions, don't hesitate to contact me for clarification.
- **Matrix, Diagonalizable**: A matrix M is **diagonalizable** if there is an invertible matrix P and a diagonal matrix E such that we can write $M = PEP^{-1}$. Remember that $n \times n$ matrices are diagonalizable if and only if a basis for $\mathbb{R}^n$ can be made out of their eigenvectors.
- **Matrix, Symmetric**: A matrix M is symmetric iff $M^T = M$. Symmetric matrices have several nice properties: namely, (1) they're diagonalizable, (2) two eigenvectors corresponding to different eigenvalues are orthogonal, and (3) they're positive-(definite/semidefinite) iff their eigenvalues are all ($>$ / $\geq$) than 0.
- **Multiplicity, Algebraic**: For a matrix M and for a given eigenvalue λ, the algebraic multiplicity is the exponent n on the term $(x - \lambda)$ in the characteristic polynomial of M, after factoring the polynomial into the form $(x - \lambda)^n p(x)$, where $p(\lambda) \neq 0$.
- **Multiplicity, Geometric**: The dimension of the eigenspace corresponding to λ. Note that the algebraic and geometric multiplicities of eigenvalues can vary wildly: see HW 6, problem 3 for an example of this.
- **Positive-definite/semidefinite**: A matrix M is positive-definite iff for all real-valued vectors x, $x^T M x > 0$; it's positive-semidefinite if $x^T M x$ is merely ≥ 0 for all such x.
- **Matrix, Digraphs associated to a**: Know how to form the digraph associated to a matrix, as covered in week 9; know also the theorem we proved there that says that the m, n-th entry in P^k is > 0 iff there is a path of length k between the nth and mth nodes in P's adjacency graph.
- **Projection**: Know how to take projections, and how to use Gram-Schmidt to orthogonalize things. Look at the recitation notes from week 6 for an in-depth explanation of how to do this.
- **Stable Vector**: For a probability matrix P, a eigenvector corresponding to the eigenvalue 1, such that the sum of its entries is 1. Recall that there is a unique stability vector if P is regular (i.e. $P > 0$,) and furthermore this stable vector will have all of its entries > 0. (However, if P is not regular,

there is no such guarantee, and for irregular probability matrices there are often multiple stable vectors.)

3. Random Questions

If you were in recitation, you saw me answer the coin problem on the infinte chessboard! If you were not in recitation, you should have been. :p If you're burningly curious, feel free to stop by or write and I can explain the basic idea, though I think some of you guys have more pressing concerns this week.