MA 1B NOTES - WEEK 1 - INTRODUCTION TO MATRICES

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1. Basic Information

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Office Hour: Sunday, 8pm.

Recitation: 2-3, Th, Downs 119.

Course Website: http://math.caltech.edu/classes/ma1b-Pr/index.html .

Policies on dropping HW and late HW have changed from last quarter, so go there and actually read the thing. Otherwise you may do something silly in the future. So, without further ado:

2. Exercises!

Question 2.1. *(Challenge) What is the largest set of vectors in $\mathbb{R}^3$ such that any three are linearly independent? (We say a collection of n vectors in $\mathbb{R}^3$ is* ***linearly independent*** *if the rank of the $n \times 3$ matrix formed by those n vectors is n.)*

Question 2.2. *What are all solutions to the system of linear equations*

$$\begin{aligned}\frac{2}{3}x + \frac{5}{3}y + 1z &= b_1\\ \frac{7}{3}x + \frac{8}{3}y + 5z &= b_2\\ -2x - 24y + 6z &= b_3\end{aligned},$$

for (b_1, b_2, b_3) equal to

a) $(1, 1, -18)$?
b) $(0, 0, 0)$?
c) $(2, 3, 4)$?
d) $(2, 7, 0)$?

Answer 2.3. (1) **Step 1: Translate into matrix notation**. To find solutions to this system of linear equations, we first write them as a matrix: specifically, the above system of linear equations has the corresponding coefficient matrix

$$\begin{pmatrix} 2/3 & 5/3 & 1 \\ 7/3 & 8/3 & 5 \\ -2 & -24 & 6 \end{pmatrix}.$$

Call this matrix A.

(2) **Step 2: Reduce to row-echelon form**. Once we have written out the coefficient matrix for our system of linear equations, we then want to reduce the matrix

$$[A|\bar{b}] := \left(\begin{array}{ccc|c} 2/3 & 5/3 & 1 & b_1 \\ 7/3 & 8/3 & 5 & b_2 \\ -2 & -24 & 6 & b_3 \end{array}\right)$$

into row-echelon form. So: specifically, in the case of a), we have that the above matrix is

$$\left(\begin{array}{ccc|c} 2/3 & 5/3 & 1 & 1 \\ 7/3 & 8/3 & 5 & 1 \\ -2 & -24 & 6 & -18 \end{array}\right);$$

pivoting on $\frac{2}{3}$ yields

$$\left(\begin{array}{ccc|c} 1 & \frac{5}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & -\frac{19}{6} & \frac{3}{2} & -\frac{5}{2} \\ 0 & -19 & 9 & -15 \end{array}\right);$$

which becomes (after pivoting on (-19/6))

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{51}{19} & -\frac{9}{19} \\ 0 & 1 & -\frac{9}{19} & \frac{15}{19} \\ 0 & 0 & 0 & 0 \end{array}\right)_{(a)}.$$

Similar work will transform the matrices for (b), (c), and (d) into

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{51}{19} & 0 \\ 0 & 1 & -\frac{9}{19} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)_{(b)}, \left(\begin{array}{ccc|c} 1 & 0 & \frac{51}{19} & 3-\frac{120}{38} \\ 0 & 1 & -\frac{9}{19} & \frac{24}{19} \\ 0 & 0 & 0 & -14 \end{array}\right)_{(c)}, \text{ and } \left(\begin{array}{ccc|c} 1 & 0 & \frac{51}{19} & 3 \\ 0 & 1 & -\frac{9}{19} & 0 \\ 0 & 0 & 0 & 6 \end{array}\right)_{(d)}.$$

(3) **Step 3: Check for consistency.** In order for solutions to even exist at all, we need that the system of equations we are checking is "consistent" – i.e. that solutions exist at all. From section 2.5, we know that a system of equations is consistent if and only if the rank of $[A]$ (i.e. the number of nonzero rows in the row-echelon reduced form of $[A]$) is equal to the rank of $[A|\bar{b}]$. So: The rank of $[A]$ is clearly 2, from our work in step 2; in cases (a) and (b), the rank of $[A|\bar{b}]$ is also 2, while in cases (c) and (d) the rank of $[A|\bar{b}]$ is 3. So no solutions are even possible for (c) and (d)! This is in accord with our intuition, as the third row in each of those matrices corresponds to the equations $0x + 0y + 0z = -14$ or 6, respectively, which is obviously unsatisfiable. So it suffices to only check (a) and (b).

(4) **Step 4: Compute the basic solution of the system $Ax = 0$.** To do this: first, note that requiring

$$\begin{pmatrix} 1 & 0 & \frac{51}{19} \\ 0 & 1 & -\frac{9}{19} \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

is equivalent to requiring that

$$\begin{aligned} 1x_1 + 0x_2 + \frac{51}{19}x_3 =& 0 \\ 0x_1 + 1x_2 - \frac{9}{19}x_3 =& 0; \end{aligned}$$

i.e. that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{51}{19} \\ -\frac{9}{19} \end{bmatrix} \cdot x_3.$$

Multiplying by 19 gives us that the generalized solution to $Ax = 0$ is

$$\begin{bmatrix} 51 \\ -9 \\ 19 \end{bmatrix} \cdot \gamma,$$

for γ a scalar. Note that this is also the family of all solutions for (b), as we had the vector $\bar{b} = (0, 0, 0)$ in that exercise.

(5) **Step 4: Find a particular solution, and use it to deduce all posible solutions.** So: we are left only with (a). To find a particular solution, simply look at $\bar{x} = [-\frac{9}{19}, \frac{15}{19}, 0]$; this is obviously *a* solution to the system

$$\begin{aligned} 1x_1 + 0x_2 + \frac{51}{19}x_3 =& 0 \\ 0x_1 + 1x_2 - \frac{9}{19}x_3 =& 0; \end{aligned}$$

(To see this: just let $x_3 = 0$.)

All possible solutions are consequently classified as of the form

$$x = \begin{bmatrix} -\frac{9}{19} \\ \frac{6}{19} \\ 0 \end{bmatrix} + \begin{bmatrix} 51 \\ -9 \\ 19 \end{bmatrix} \cdot \gamma,$$

for γ a scalar.

Remark 2.4. Why are all solutions of this form? First: notice that any vector in the above form *is* a solution, because (letting $x = (6\gamma, 9\gamma, 19\gamma)^T, y = (-\frac{9}{19}, \frac{6}{19}, 0)^T$)

$$A \cdot (x + y) = A \cdot x + A \cdot y = 0 + (1, 1, -18)^T.$$

Conversely, pick any solution z to $Az = (1, 1, -18)^T$; we then know that (for y as above)

$$A \cdot (z - y) = A \cdot z - A \cdot y = (1, 1, -18)^T - (1, 1, -18)^T = 0,$$

and thus that we can write z as the sum of a solution to $Ax = 0$ (specifically, $x = z - y$) and y. So any solution can be expressed in the form above – so we have indeed classified all solutions!