MATH 1A, SECTION 1, WEEK 8 - RECITATION NOTES

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Abstract. These are the notes from Thursday, Nov. 19th's recitation, on the exponential and logarithmic functions. We open with a review of the properties of the logarithmic function, and work some examples of integration and logarithmic differentiation; from there, we shift to discussing the exponential function, quickly reviewing its properties and proving that it is "unique" among functions that share these qualities.

1. Homework comments

- Homework average: around 85%.
- Comments: People did mostly OK on this set. A few points that people should avoid:
  - Don't simply calculate a decimal approximation to an equation and use this as your answer! Often, this means that you lose a lot of valuable information; for example, the algebraic solution to $0 = x^2 - 2ex + e^2$ is the relatively elegant $\sqrt{e}$, whereas a numerical approximation is just 1.649, which looks like nothing remotely interesting.
  - So: last week I said that you didn't have to simplify things if they didn't have nice closed forms. I'd like to make the obverse remark here: if something has a beautiful closed form, please simplify it! The answer
  $$\frac{1}{1+\frac{1}{1+\frac{1}{1+1/e}}}$$
  is not nearly as illuminating as
  $$\frac{2e+1}{3e+2}.$$
  - Be careful with your statements of theorems and definitions! I.e. double-check your proofs to make sure that they're actually what the definitions and theorems in your text say they are; several people lost points this set because they started a problem with an incorrect definition or statement of a theorem.
  - Finally, (as always,) make sure to use lots of words in your proofs! Words are your friends.

2. Random Question

So: if you give me a set S, I can define a **coloring** of S by simply assigning a color to every element of S. For example, we can define a coloring of the set of points on the unit circle by saying that every point that has positive y-coördinate is colored red, and that all of the other points are blue. Similarly, we can define a coloring of the students at Caltech by coloring people according to house affiliation – Fleming House students would be colored red, Blacker House students would be colored black, and so on/so forth (At least, I think all of of your houses have associated colors.)

The question, then, is the following:

Question 2.1. *Can you find a way of coloring the entire plane $\mathbb{R}^2$ with the three colors {red, green, blue}, so that every two points that are distance 1 away from each other are different colors?*

3. The Logarithm - Basic Properties

So: we review here a few of the basic properties of the natural logarithm.

Definition 3.1. We denote the natural logarithm of x, for all $x > 0$, as the value of the integral

$$\int_1^x \frac{1}{t} dt,$$

and denote this quantity as $\log(x)$. (Note here that we assume all of our logarithms are in base e unless explicitly defined otherwise – this is in sharp contrast to other fields like physics (where most logarithms are base 10) or computer science (where most logarithms are base 2). Wikipedia has a nice little discussion about this issue, if you're interested.)

The function $\log(x)$ has the following remarkable properties:

- $\log(xy) = \log(x) + \log(y)$.
- $\log(x^y) = y \log(x)$.
- $\log(1) = 0$.
- $\log(x)$ is a bijection (see the notes from Week 2 if you've forgotten what a bijection is!) from $(0, \infty)$ to $\mathbb{R}$.
- $\frac{d}{dx}(\log(x)) = \frac{1}{x}$ (this is immediate from the Fundamental Theorem of Calculus.)

The proofs of most of the above are in your text and were done in class; feel free to write if you would like to see another proof of any of these properties!

So: to get a feel for how $\log(x)$ works, let's work a pair of sample integrations with it:

Example 3.2. Calculate

$$\int \log(x) dx.$$

Proof. We use integration by parts here, and set

$$\begin{array}{ll} u = \log(x) & du = \frac{1}{x} dx \\ dv = 1dx & v = x \end{array}$$

(Why did we make these choices? Again, as we mentioned in week 7's notes, we always want to choose our u and v to make our integrals simpler! In this case, $\log(x)$ is something which becomes much simpler after differentiation – it's just a power of x – so we put it in the u-slot and put what's left over – $1dx$ – in the dv-slot.)

This then tells us that

$$\begin{aligned}\int \log(x)dx &= x \cdot \log(x) - \int \frac{1}{x} \cdot x dx \\ &= x\log(x) + \int dx \\ &= x\log(x) + x + C,\end{aligned}$$

where C is a constant of integration. □

Example 3.3. Calculate

$$\int \tan(x)dx.$$

Proof. We use a substitution here: explicitly, let $u = \cos(x)$. Then $du = -\sin(x)dx$, and we have that

$$\begin{aligned}\int \tan(x)dx &= \int \frac{\sin(x)}{\cos(x)}dx \\ &= \int -\frac{1}{u}du \\ &= -\log(|u|) + C \\ &= -\log(|\cos(x)|) + C.\end{aligned}$$

□

Remark 3.4. So: above, we just wrote that

$$\int \frac{1}{t}dt = \ln(|t|) + C.$$

Why is this true? I.e. why did we put absolute values there? This bears a little discussion, as it turns out that having those absolute value signs is really crucial.

So: first, recall the definition of what an indefinite integral *is* – i.e. if we write

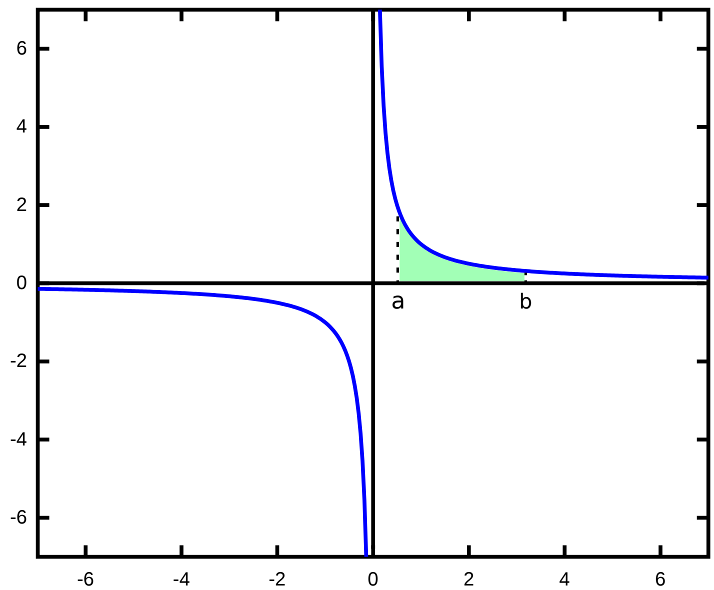
$$\int f(t)dt = F(t),$$

this just means that we consider $F(t)$ to be the antiderivative of $f(t)$ – i.e. that $F'(t) = f(t)$, or equivalently (by the FTC)

$$\int_a^b f(t)dt = F(b) - F(a),$$

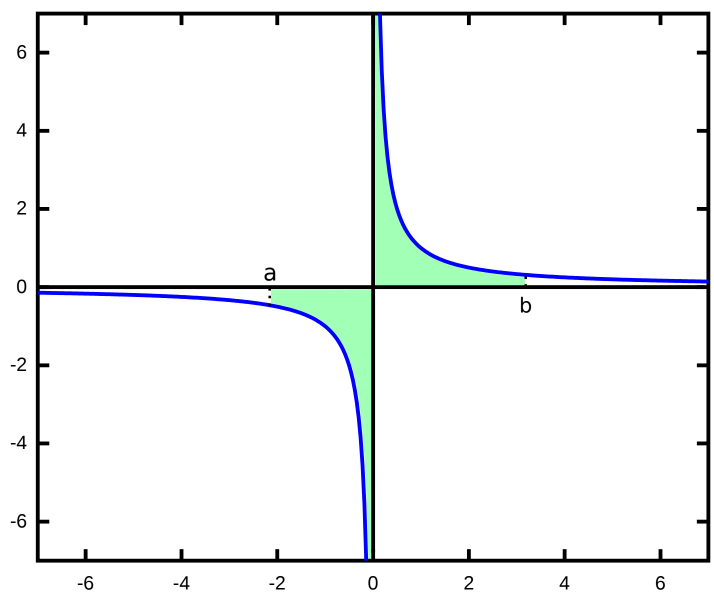
wherever the integral on the left exists.

So: where does this integral exist? Well, if both a and b are positive, it exists, because it's just the area under the curve



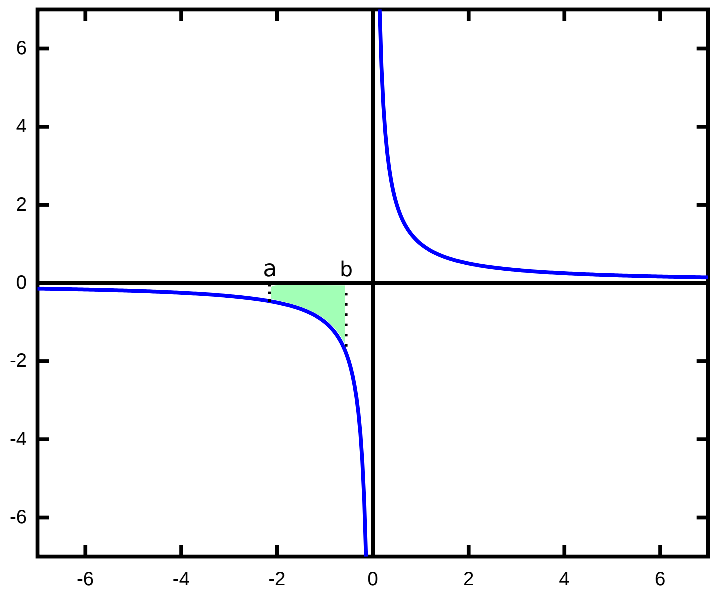
We know that this is just $\ln(b) - \ln(a)$ – so no need for absolute values yet.

If a is negative and b is positive, then this integral is the area under the curve

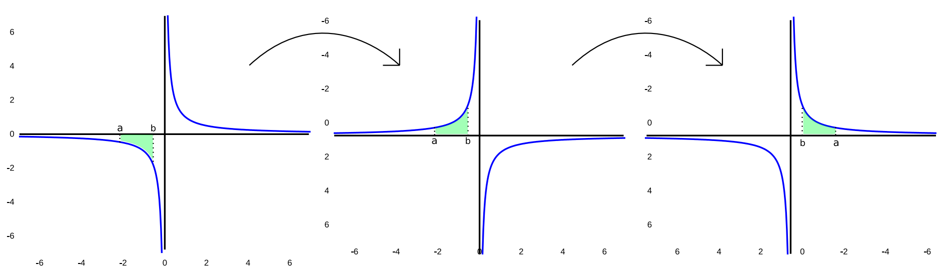


But this area can't be computed! I.e. you wind up trying to subtract negative infinity from infinity, which just doesn't work at all. So, for whatever function we decide that the antiderivative $\int 1/t$ is, we don't have to worry about it making sense if a is negative and b is positive.

Finally, we consider what happens when both a and b are negative. In this case, our integral is the signed area under the curve below:



This area can be computed! in specific, if we flip first the x and then the y axes, we can see that the unsigned area here is just $\ln(|a|) - \ln(|b|)$!



Consequently, the signed area is $-1 \cdot (\ln(|a) - \ln(|b|)) = \ln(|b|) - \ln(|a|)$, because $1/t$ is negative when $t < 0$. Thus, our antiderivative has to be $\ln(|t|) + C$ in order for all of this to work out.

4. The Logarithm and Derivatives

One useful thing that the logarithm allows us to do is to calculate the derivatives of functions that we would otherwise have no techniques to attack. This method is best illustrated by an example:

Example 4.1. Calculate

$$\frac{d}{dx}x^x.$$

Proof. We first note, as a warning, that techniques like the power rule ($(x^n)' = nx^{n-1}$) are completely useless here, as it only works for x raised to some constant power – not a variable!

So we must do something clever – namely, we must use logarithmic differentiation. Explicitly, what do we do? First, define the functions

$$\begin{aligned} f(x) &= x^x, \\ g(x) &= \log(f(x)) \end{aligned}$$

Then, we have that (via the chain rule)

$$\begin{aligned} g'(x) &= (\log(f(x))' \\ &= f'(x) \cdot \frac{1}{f(x)} \end{aligned}$$

and thus that (solving for $f'(x)$)

$$f'(x) = f(x) \cdot g'(x).$$

But we can actually calculate $g'(x)$! Explicitly, it's given by

$$\begin{aligned} g'(x) &= (\log(f(x))' \\ &= (\log(x^x))' \\ &= (x\log(x))' \\ &= x \cdot \frac{1}{x} + \log(x) \\ &= 1 + \log(x); \end{aligned}$$

consequently, we have that

$$\begin{aligned} f'(x) &= f(x) \cdot g'(x) \\ &= x^x \cdot (1 + \log(x)). \end{aligned}$$

Note that the entire derivation of $f'(x) = f(x) \cdot g'(x)$ we did above had nothing to do with our choice of $f(x)$! In other words, we can use this process to find derivatives of all kinds of functions f for which we can differentiate their logarithms! $\square$

We do one more example to further illustrate the technique:

Example 4.2. Calculate

$$\frac{d}{dx}\left(x^{e^x}\right).$$

Proof. As before, let

$$\begin{aligned} f(x) &= x^{e^x}, \\ g(x) &= \log\left(x^{e^x}\right) \\ &= e^x \cdot \log(x). \end{aligned}$$

Then, just as before, the chain rule tells us that

$$f'(x) = g'(x) \cdot f(x);$$

thus, because

$$\begin{aligned} g'(x) &= (e^x \cdot \log(x))' \\ &= e^x \cdot \frac{1}{x} + e^x \cdot \log(x), \end{aligned}$$

we have that

$$f'(x) = x^{e^x} \cdot \left(e^x \cdot \frac{1}{x} + e^x \cdot \log(x) \right).$$

□

5. The Exponential Function - Basic Properties

The above discussion about the logarithm leads itself to a discussion of its inverse function – the exponential function, e^x.

Again, note the following list of useful and remarkable properties:

- e is the unique number such that $\log(e) = 1$.
- $e^{xy} = e^x e^y$.
- $(e^x)' = e^x$.
- $\int e^x = e^x + C$.

6. The Exponential Function - A Sketch Proof of its Uniqueness

We now make a rather remarkably bold claim: that the first two properties listed above uniquely define the exponential function among all differentiable functions! To put this more explicitly:

Theorem 6.1. *Suppose that f is a differentiable function that satisfies just the following two properties:*

- $f(1) = e$.
- $f(x + y) = f(x)f(y)$, *for any* $x, y \in \mathbb{R}$.

Then $f(x) = e^x$.

Proof. So: where do we start? Well – the property $f(x + y) = f(x)f(y)$ seems like something that will be interesting to play with; so let's see what we can get from that. This says that, essentially, this function transforms addition into multiplication – so what happens when we put the additive identity, 0, into the function? We'd expect to get the multiplicative identity – and indeed, because

$$\begin{aligned} f(0) &= f(0 + 0) = f(0)f(0) \\ \Rightarrow & f(0) = 1 \text{ or } f(0) = 0. \end{aligned}$$

But if $f(0) = 0$, then $f(1) = f(0 + 1) = f(0)f(1) = 0 \cdot e = 0$ which contradicts our second property – so $f(0) = 1$! I.e. it sends the additive identity to the multiplicative identity.

As well, this property is interesting from a derivative-point of view, in that it tells us that (by using the definition of the derivative and the above realization that $f(0) = 1$)

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= f(x) \cdot f'(0). \end{aligned}$$

So: in other words, the derivative of f at any point is just a constant ($f'(0)$) times the value of f at that point! This seems very e^x-like, and is quite promising.

In fact, this actually is the last nail in the coffin! For we can now show that our function f is e^x. To do this – we engage in one little piece of trickery. Namely: suppose something slightly weaker, that $f(x)$ is equal to e^{cx} for some constant c. Then, we'd have that $f(x) \cdot \frac{1}{e^{cx}}$ is identically 1, right? So it suffices to show that $f(x) \cdot \frac{1}{e^{cx}}$ is a constant that's equal to 1. But how can we show that something is a constant? By taking its derivative! So: calculation tells us that

$$\begin{aligned} \left(f(x) \cdot \frac{1}{e^{cx}} \right)' &= f'(x) \cdot \frac{1}{e^{cx}} + f(x) \cdot -c \cdot \frac{1}{e^{cx}} \\ &= f'(0) \cdot f(x) \cdot \frac{1}{e^{x}} - c \cdot f(x) \cdot \frac{1}{e^{cx}}. \end{aligned}$$

But because $f'(0) = c \cdot e^{0 \cdot c} = c$, we know that $c = f'(0)$ – and thus that the above equation is zero! So $f(x) \cdot \frac{1}{e^{cx}}$ is a constant, and at 0 we know that

$$f(0) \cdot \frac{1}{e^{c \cdot 0}} = 1 \cdot 1 = 1.$$

So it's a constant equal to 1 – so $f(x) = e^{cx}$! And because $f(1) = e = e^1$, we know that c must be equal to 1; thus, at last, we have that

$$f(x) = e^x$$

as claimed. □

Cool, no?