MATH 1A, SECTION 1, WEEK 6 - RECITATION NOTES

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Abstract. These are the notes from Thursday, Nov. 5th's recitation, which comes in two main parts. The first is a discussion of the issues that came up on the midterm; specifically, we talk about things like notational issues, commonly made mistakes, and stylistic problems. After finishing that, we move to working further with derivatives, looking at a few "pathological" examples: here, we demonstrate just how much mathematical power is contained within the relatively few concepts we have thus far.

1. Midterm comments

- Midterm average: around 67%.
- Midterm setup: I graded questions 2 and 3, Prof. Ryckman graded questions 1 and 4.
- Solutions are up on the website, with nicely detailed writeups for all of the questions. If you're confused about some of the midterm questions, write me and I'll be glad to clarify what I've put up there further.

So: there were a decent number of pitfalls that people wandered into in this test. Here's a list of some of the most frequently occuring problems on the midterm:

1.1. $\Leftrightarrow$ versus $=$.

$\Leftrightarrow$ means "is equivalent to." $=$ means "is equal to."

Whenever you use either symbol in your work, make a point of translating it over into words and making sure that what you've written isn't nonsense. I.e. you should **never** write

$$x^3 + 3x^2 + 3x + 1 \Leftrightarrow (x+1)^3,$$

because that doesn't make any sense! What would it even mean to say that two formulas were equivalent? Instead, what you mean is to write

$$x^3 + 3x^2 + 3x + 1 = (x+1)^3,$$

because your assertion here is that these two objects are **equal**.

Similarly, **don't write** that

$$\begin{aligned} &-1 \leq \sin(x) \leq 1 \\ =&-|x| \leq |x|\sin(x) \leq |x|, \end{aligned}$$

because again this makes no sense! – what would it even mean to say that two equations are equal? Instead, what should be written is

$$\begin{aligned} &-1 \leq \sin(x) \leq 1 \\ \Leftrightarrow&-|x| \leq |x|\sin(x) \leq |x|, \end{aligned}$$

because you're clearly asserting that these two equations are **equivalent** – i.e. that the first equation holds if and only if the second equation holds.

If this is in any way unclear, please write me and I can give you lots of examples to hopefully clarify this further.

1.2. **Induction.** Know how to write an inductive proof! Inductive proofs follow a very carefully set pattern:

- First, notify the reader that you're going to use induction to prove your claim, by saying something like "We proceed by induction."
- Begin by proving the **base case**. Explicitly state that you're proving the base case, and state what the base case is – i.e. tell us which value of n you're starting on. If you want to prove that a proposition holds for all $n \geq 1$, your base case must begin at 1!
- After you prove the base case, move to the **induction step**. State what your induction hypothesis is – i.e. state that you're assuming your proposition holds for some value of n. Then, prove that this claim holds for $n+1$. Make sure to identify where in the proof you use your induction hypothesis; this is helpful to the reader.

1.3. $\epsilon - \delta$**-proofs.** So: how does an $\epsilon - \delta$ proof of a limit work? I.e. suppose that we're trying to show that

$$\lim_{x \to c} f(x) = A.$$

What do we do? Well, if we're trying to show this by the $\epsilon - \delta$ definition, we do the following:

- We start by considering **any** possible $\epsilon > 0$. We don't get to pick this ϵ; it could be anything!
- We then define some constant $\delta > 0$. We can only define δ in terms of constants we know – so we could do something like setting $\delta = \epsilon/2$, or $\delta = \pi/4$, or anything like that. We cannot, however, use a variable we haven't defined in our definition of δ! I.e. saying that $\delta = x$ is not remotely allowed.
- Now, we consider any possible x such that $0 < |x - c| < \delta$. We then use everything that we know to try to show that $|f(x) - A|$ has to be less than ϵ – if we can do this, then we're done with the proof. That's it!

I really want to emphasize the parts about considering all possible values of ϵ and x, and not picking δ so that it depends on x – these are mistakes that will really mess up your proof if you make them, and they're somewhat subtle points. Again, if you're confused, write me and I can send you lots more examples and verbiage.

1.4. **Limits and Multiplication.** So: question: when is

$$\lim_{x \to c} f(x)g(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)?$$

Answer: Only when both of the limits $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist!

We repeat and bold this for emphasis: **The limit of a product is equal to the product of the limits only if both individual limits exist!** This is a really critical point, and cannot be hammered home enough.

2. Derivatives - Some Unexpected Constructions

In a previous recitation, we mentioned that there were functions that were continuous at **only one point** in $\mathbb{R}$ – an example we gave was

$$g(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

A similar question can be asked for derivatives: is there a function that's **differentiable** at only one point in $\mathbb{R}$? It turns out that there is:

Proposition 2.1. *The function*

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ \sin(x), & x \notin \mathbb{Q} \end{cases}$$

is differentiable at exactly one point – specifically, the origin.

Proof. So: we know trivially that $f(x)$ is discontinuous at every point $p \neq 0$, as

$$\lim_{x \to p, x \in \mathbb{Q}} f(x) = \lim_{x \to p, x \in \mathbb{Q}} x = p$$

and

$$\lim_{x \to p, x \notin \mathbb{Q}} f(x) = \lim_{x \to p, x \notin \mathbb{Q}} x = \sin(p)$$

are distinct quantities for any $p \neq 0$. Because a function must be continuous at a point in order to be differentiable at a point (from class,) we then know that f is not differentiable at any point $p \neq 0$.

So we just need to show that f is differentiable at 0 – i.e. that the limit

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}$$

exists.

To see this, we use the squeeze theorem. First, recall from the proof that $\lim_{x\ to 0} \sin(x)/x = 1$ the inequality

$$\cos(h) \leq \frac{\sin(h)}{h} \leq \frac{1}{\cos(h)}$$

for all h in some small neighborhood of 0.

As well, because cos is bounded above by 1 and positive in some small neighborhood of 0, we have as well that

$$\cos(h) \leq \frac{h}{h} = 1 \leq \frac{1}{\cos(h)}.$$

Combining, this gives us that

$$\cos(h) \leq \frac{f(h)}{h} \leq \frac{1}{\cos(h)}$$

in some neighborhood of zero. As both $\cos(h)$ and $1/\cos(h)$ both go to 1 as h goes to zero, we can apply the squeeze theorem to see that

$$f'(h) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h} = 1$$

as well. Thus, f is a function that is differentiable at exactly one point! □

(At this point in the class, a vote was taken on whether to work through several examples of the chain and product rule, or to illustrate a particularly fascinating counterexample in analysis. The counterexample handily won the vote: if you feel cheated, and want to see a series of worked examples of derivatives, this website has a list of carefully-worked examples.)

(Also: the following is really cool, but really complicated! Don't worry about things being confusing; this material will not reappear later in the class.)

So: here's a question:

Question 2.2. *Is there a function $f(x)$ such that*

- *$f(x)$ is continuous on all of $\mathbb{R}$,*
- *wherever $f'(x)$ exists, it is 0, and*
- *$f(0) = 0, f(1) = 1$?*

To answer this question, we need to quickly develop two concepts:

2.1. **Binary and Ternary Numbers.** So: on a day-to-day basis, most of use constantly use the **decimal number system** – i.e. "base 10". What does this mean? Well, it means that we "count" using a system of ten digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ – i.e. that when we see a number like

$$385.23$$

we know that this stands for the quantity

$$3 \cdot 10^2 + 8 \cdot 10^1 + 5 \cdot 10^0 + 2 \cdot 10^{-1} + 3 \cdot 10^{-2}.$$

If you think back to your elementary school days, you'll find that this concept was being ingrained in your minds even then – the "tens place," "hundreds place," and so on/so forth were just a shorthand for the powers of ten we wrote above.

But (apart from the physiologically convenient feature that we have ten fingers) there's really nothing special about using ten digits – we could make a number system using any number of digits! And, in fact, people have – those of you familar with computer programming will be familiar with **binary**, which is a system of counting that uses only **two** digits, 0 and 1.

Explicitly, in binary, we write all of our numbers in the form

$$1011.01$$

where this stands for the real number

$$1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2}.$$

But (again, apart from the physiologically convenient feature that we have two hands: cf. the late Claude Lévi-Strauss) there's nothing special about 2, either! We could make a system with three digits – $\{0, 1, 2\}$, and declare that when we write a number like

$$2211.02$$

we intend that this stands for the real number

$$2 \cdot 3^3 + 2 \cdot 3^2 + 1 \cdot 3^1 + 1 \cdot 3^0 + 0 \cdot 3^{-1} + 2 \cdot 3^{-2}.$$

This system is called the **ternary** number system; it comes up far less often than either the decimal or binary number systems, but is quite useful in its own right, as we will soon demonstrate.

2.2. **The Cantor Set.** The **Cantor set** is a **fractal** which we define as follows:

- Start with the interval $[0, 1]$.
- Remove the open middle third $(1/3, 2/3)$ from this interval.
- Take your two remaining intervals $[0, 1/3]$ and $[2/3, 1]$.
- Remove the open middle thirds from each of **these** intervals – i.e. delete $(1/9, 2/9)$ and $(7/9, 8/9)$ from each set.
- Take the remaining four intervals.
- Remove *their* middle thirds.
- Repeat.

The set that you get at the end of this process – i.e. after you repeat this process "infinitely many times" – is called the Cantor set. A picture of what this process is doing is attached below:

The Cantor set has a number of amazing properties, which you can read about at length on Wikipedia. Its most salient property for what we're doing here, however, we describe here:

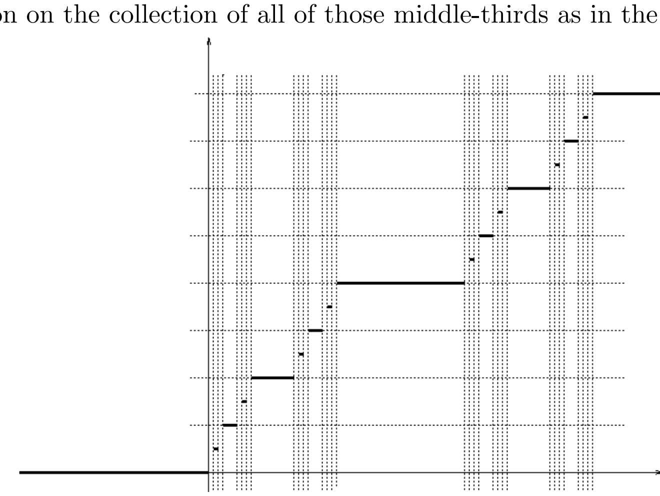
Proposition 2.3. *The Cantor set can be written as the set*

$$C = \{x \in [0, 1] : x \text{ can be written in ternary using only 0's and 2's}\}.$$

2.3. **Our Function.** So, we return to our original goal: to construct a function such that

- $f(x)$ is continuous on all of $\mathbb{R}$,
- wherever $f'(x)$ exists, it is 0, and
- $f(0) = 0, f(1) = 1$.

How can we do this? Well, it turns out that we can "make" a function which does this by using the Cantor set! I.e. what we do is we "take" the collection of all of the middle-thirds that we popped out of the Cantor set, and define a step function on the collection of all of those middle-thirds as in the picture below:



Explicitly, this function is defined as follows:

- Take a number $a \in [0, 1]$.
- Write a in ternary, using no 1's if it's possible – i.e find a string $.a_0a_1a_2a_3\ldots$ such that $a = .a_0a_1a_2a_3\ldots$ in ternary.
- Define the binary number $.y_0y_1y_2y_3\ldots$ as follows:
  - if the string $.a_0a_1a_2a_3\ldots$ has no 1's in it, define $y_i = 1$ if $a_i = 2$ and $y_i = 0$ if $a_i = 0$ for every i. This is your number.
  - if the string $.a_0a_1a_2a_3\ldots$ has a 1 in it – let k be the smallest number so that $a_k = 1$. Let $y_i = 0$ for every $i > k$, $y_i = 0$ if $i \leq k$ and $a_i = 0$, and $y_i = 1$ if $i \leq k$ and $a_i \neq 0$. This is your number.
- Let $y = .y_0y_1y_2y_3\ldots$.
- Define $f(a) = y$.

The explicit definition is kind of ugly; it's better to focus on the picture, as it makes it a lot clearer that the function is in fact continuous, has zero derivative wherever it has a derivative, and goes from 0 to 1!

Again, if this was confusing, don't worry; this was a really high-level example that I wanted to show you all because it's so beautiful. If you can take anything away from this discussion – an idea of how binary works, the mental image for how the Cantor set is constructed – you've done quite well.

(If you're interested in a slightly different writeup, Wikipedia has a good discussion of the function, and is where I pulled the picture from.)