MATH 1A, SECTION 1, WEEK 5 - RECITATION NOTES

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ABSTRACT. These are the notes from Thursday, Oct. 29's recitation. We open with a quick review of the course thus far, work some problems that illustrate the techniques we've developed so far, and then transition to a discussion of the derivative, working several examples to make their calculations clear.

1. HW COMMENTS

Average: 82%. Pretty good, all in all. Specific comments: just two minor-yet-key points.

- A number of people got the formula for a volume of revolution wrong. Specifically, on some interval $[a, b]$, take the region above the function $g(x)$ and below the function $f(x)$. If we rotate this region around the x-axis, we get a shape. What is its volume? As many of you noted, it's given by the formula
$$\int_a^b \pi(f(x))^2 - (g(x))^2)dx.$$
However, some people were a bit confused and had the formula
$$\int_a^b \pi(f(x) - g(x))^2 dx,$$
which doesn't work at all.
It's easy to confuse the two formulas, so if you're ever confused, simply think about what you're trying to do. To measure the volume of this shape bounded above by f and below by g, you're just taking the volume of the shape given by rotating f and "removing" the volume of the shape given by rotating g. So you would expect to see $\int_a^b \pi(f(x))^2 - (g(x))^2)dx.2$ as your term; as by applying linearity, we can split this integral into the volume bounded by f minus the volume bounded by g, as we just discussed.
- If we have the inequality
$$-M \leq g(x) \leq M,$$
when do we have that
$$-Mf(x) \leq g(x)f(x) \leq Mf(x)?$$
Only when f is positive! Be careful with this when you're applying squeeze-theorem-like techinques to limits, and if you're not sure just put absolute value signs around the $f(x)$, like so:
$$-M|f(x)| \leq g(x)|f(x)| \leq M|f(x)|$$
will always hold whenever $-M \leq g(x) \leq M$ is true.

2. Review of the Class Thus Far

2.1. **Mathematical Reasoning.** We opened our course with arguably the hardest concept you'll have to master in your mathematics courses here at Caltech – the ideas of mathematical reasoning and proof. Explicitly, we discussed the following concepts:

- The concept of "proof," both in the abstract sense (a proof of some claim X is an argument which takes a series of assumed truths, and demonstrates that X must follow from them) and in the concrete sense (how do we write a proof? When do we use symbols like $\Leftrightarrow$ or $\Rightarrow$, or $\forall$ or $\exists$?)
- The methods of proof. Specifically, how to do a proof by induction and a proof by contradiction; what language to use, what formats to follow.

2.2. **Integration.** With an understanding of what it means to prove things, we moved onto the concept of integration. Integration, as we dealt with it in this course, involved understanding the following things:

- Step functions. Specifically, their definitions and their integrals.
- The definition of integrability – what it means for a function to be integrable, and several sufficient conditions for integrability (a function is integrable if it is piecewise continuous or piecewise monotone, for example.)
- Methods of integration: know how to use the linearity of the integral and how to scale an integral.
- Previously known integrals: note that we know how to integrate polynomials and trigonometric functions.

2.3. **Limits.** Finally, we changed tacks somewhat to study limits and their properties, which we quickly list below:

- The definition of a limit. We discussed the "intuitive" meaning of a limit, as well as the "neighborhood" definition and the $\epsilon-\delta$ definition, and discussed how to use these definitions to "prove" that a limit does or does not exist.
- Properties of the limit: note that we can multiply, add, and subtract limits from each other to get limits. Also note that we have the squeeze/two-policeman theorem, which is invaluable in calculations.
- Previously calculated limits – we've calculated several limits that are often handy: amongst these are
$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1, \lim_{x\to 0} \sin(1/x)\ DNE, \lim_{x\to 0^+} \frac{|x|}{x} = 1, \lim_{x\to 0} \frac{1}{x}\ DNE.$$
Know these results, even if you're not sure how to recreate their proofs.
- Intermediate Value Theorem – know its statement (i.e. that on a closed interval, a continuous function takes on every value between its maximum and its minimum) and how to use it.

3. Review Questions

If you understand all of these questions, you'll be fine on the midterm and will ace the test! We work two example problems here to illustrate the techniques we've learned so far in this course:

Question 3.1. *Let $f(x)$ be defined as*

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ x & x \notin \mathbb{Q} \end{cases}$$

Show that f is continuous at 0 and discontinuous everywhere else.

Proof. First, let's show that f is continuous at 0 – i.e. that

$$\lim_{x \to 0} f(x) = f(0) = 0.$$

(as this is the definition of continuity.)

So: trivially, we know that

$$\forall x, -|x| \leq x \leq |x|$$

and that

$$\forall x, -|x| \leq 0 \leq |x|.$$

Combining these two inequalities, we have in fact that

$$\forall x, -|x| \leq f(x) \leq |x|$$

as $f(x) = 0$ or x everywhere. Thus, because

$$\lim_{x\ to 0} -|x| = \lim_{x \to 0} |x| = 0,$$

we can apply the squeeze theorem to conclude that

$$\lim_{x\ to 0} f(x) = 0 = f(0),$$

as we claimed.

Now, we want to show that $f(x)$ is discontinuous at every $p \in \mathbb{R}$, $p \neq 0$. We actually make a stronger claim – that

$$\lim_{x \to p} f(x)\ DNE$$

for every $p \neq 0$. (this trivially implies that f is discontinuous at all such p, as to be continuous at a point a function definitely needs to at least have a limit at that point.)

So: we proceed with a proof by contradiction. Suppose not, that there is a point $p \neq 0 \in \mathbb{R}$ and a value A such that

$$\lim_{x \to p} f(x) = A.$$

Then, by the $\epsilon - \delta$ definition, we know that there is for every $\epsilon > 0$ a $\delta > 0$ such that whenever

$$0 < |x - p| < \delta,$$

we have that

$$0 < |f(x) - A| < \epsilon.$$

So, if $\epsilon = |p|/2$, we know by the definition above that there is some value δ that makes the $\epsilon - \delta$ definition work. Pick a rational point r and an irrational point s such that

$$0 < |r - p| < \min\{\delta, |p|/2\} < |s - p| < \min\{\delta, |p|/2\}$$

we can do this because rational and irrational numbers are dense in the real numbers.

But then we have that

$$0 < |f(r) - A| = |0 - A| = A < \epsilon = p/2$$

and

$$0 < |f(s) - A| = |s - A| < \epsilon = p/2$$

and

$$0 < |s - p| < p/2,$$

from the $\epsilon - \delta$ definition of the limit and our choice of s.

The first inequality means that A is more than $p/2$ away from p; but the second and third inequalities combined mean that A is less than $p/2$ away from p. These are clearly mutually impossible; so we've arrived at a contradiction. Thus we conclude that no limit can exist for f whenever $p \neq 0$. □

Question 3.2. *We claim that the integral of*

$$\int_0^p 2^{[x]} dx = 2^p - 1,$$

for any positive integer p.

Proof. So: think of the function $2^{[x]}$ as a step function that takes the values 2^{n-1} on the interval $[n, n+1)$. Then, by the definition of an integral for step functions, we have that

$$\begin{aligned}
\int_0^p 2^{[x]} dx &= \int_0^1 2^{[x]} dx + \int_1^2 2^{[x]} dx + \ldots + \int_{p-1}^p 2^{[x]} dx \\
&= (1-0) \cdot 2^0 + (2-1) \cdot 2^1 + \ldots (p - (p-1)) 2^{p-1} \\
&= \sum_{i=1}^p 2^{i-1}.
\end{aligned}$$

So: we claim that this final sum, $\sum_{i=1}^p 2^{i-1}$, is equal to $2^p - 1$. To see this, we proceed by induction. The base case is immediate, as

$$\sum_{i=1}^1 2^{i-1} = 2^0 = 1 = 2 - 1 = 2^1 - 1.$$

So: we now proceed to the inductive step. Assume that

$$\sum_{i=1}^p 2^{i-1} = 2^p - 1.$$

We then want to show that this holds for $p+1$, i.e. that

$$\sum_{i=1}^{p+1} 2^{i-1} = 2^{p+1} - 1.$$

But if we look at the left hand side of the above, we have

$$\begin{aligned}\sum_{i=1}^{p+1} 2^{i-1} &= 2^p + \sum_{i=1}^{p} 2^{i-1} \\ &= 2^p + 2^p - 1 \\ &= 2(2^p) - 1 \\ &= 2^{p+1} - 1.\end{aligned}$$

Thus, we 're done by induction. ☐

4. The Derivative - Definition

So: derivatives! We skip our usual excursion into the "why" of derivatives before in favor of getting our hands dirty with some honest calculations; next week I'll say a little more about why these are interesting. For right now, we content ourselves with merely the definition and two stupidly useful theorems:

Definition 4.1. For a function f, we define its derivative at the point x to be the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

and denote this quantity by $(f(h))'$.

Theorem 4.2. *(Product Rule) For two function $f(x), g(x)$, we have that*

$$(f(x)g(x))' = f(x)g'(x) + f'(x)g(x).$$

Theorem 4.3. *(Chain Rule) For two function $f(x), g(x)$, we have that*

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

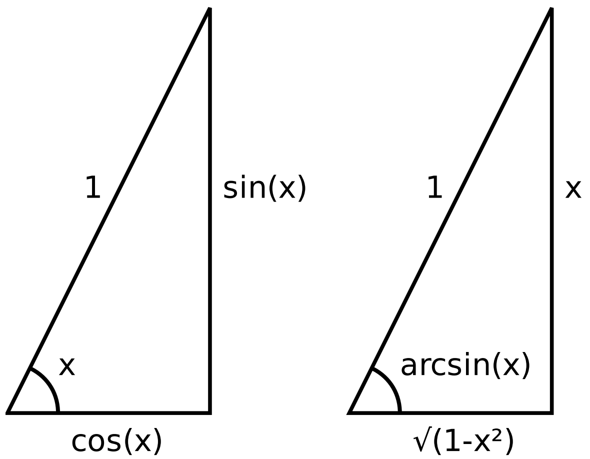
5. The Derivative - Example Calculations

Question 5.1. *We claim that the derivative of* $\arcsin(x)$ *is given by* $\frac{1}{\sqrt{1-x^2}}$.

Proof. So: we proceed by being somewhat clever, the chain rule, and drawing two triangles.

$$\begin{aligned}&\arcsin(x) = \arcsin(x) \\ \Rightarrow &\sin(\arcsin(x)) = \sin(\arcsin(x)) \\ \Rightarrow &\sin(\arcsin(x)) = x \\ \Rightarrow &\sin(\arcsin(x))' = (x)' \\ \Rightarrow &\cos(\arcsin(x)) \cdot (\arcsin(x))' = 1 \\ \Rightarrow &(\arcsin(x))' = \frac{1}{\cos(\arcsin(x))} \\ \Rightarrow &(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}},\end{aligned}$$

where we justified all of the steps up to the last with the chain rule and knowing how derivatives work, and justify our last step (setting $\cos(\arcsin(x)) = \sqrt{1-x^2}$) by looking at the definition of sin and cos via right triangles:



□

Question 5.2. *We claim that the derivative of x^n is nx^{n-1}.*

Proof. So, we first note a really useful property called the binomial theorem:

Theorem 5.3.

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n \\ &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k,\end{aligned}$$

where the symbols $\binom{n}{k}$ are integers defined by

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!},$$

where by ! we mean the factorial function, which is defined on all of $\mathbb{Z}^+ \cup \{0\}$ by

$$\begin{aligned} n! = 1 \cdot 2 \cdot 3 \cdot \ldots n, n \geq 1 \\ 0! = 1. \end{aligned}$$

Yeah – if you haven't seen it before, the above probably looks like nonsense. The important thing to know for our proof here, at least, is that the above means that

$$(x+h)^n = x^n + n \cdot x^{n-1}h + (\text{a bunch of things with } h^2 \text{ as a factor});$$

if you wanted to prove just that part, you could do this pretty easily by induction (it's certainly true for $n = 1, 2$, and you know how to multiply polynomials together; it would just be a matter of tracking down the symbols.)

The upshot of this is that once you've persuaded yourself of this fact, our proof is really really easy! Because if we can write $(x+h)^n$ in the way we discussed above, then we have that

$$\begin{aligned}
(x^n)' &= \lim_{h\ to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\ to 0} \frac{x^n + n \cdot x^{n-1}h + (\text{a bunch of things with } h^2 \text{ as a factor}) - x^n}{h} \\
&= \lim_{h\ to 0} \frac{n \cdot x^{n-1}h + (\text{a bunch of things with } h^2 \text{ as a factor})}{h} \\
&= \lim_{h\ to 0} n \cdot x^{n-1} + (\text{a bunch of things with } h \text{ as a factor}) \\
&= \lim_{h\ to 0} n \cdot x^{n-1} + \lim_{h \to 0} (\text{a bunch of things with } h \text{ as a factor}) \\
&= nx^{n-1} + 0 \\
&= nx^{n-1}.
\end{aligned}$$

$\square$

We'll work many more examples next week, as well as discuss some more complicated things we can do with derivatives.