MATH 1A, SECTION 1, WEEK 4 - RECITATION NOTES

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Abstract. These are the notes from Thursday, Oct. 22's recitation on limits; here, we try to define the concept of a limit, discuss several tools we have for calculating limits, and try our hand at several examples. (The material covered at the start of the class has been merged into the previous notes on integrals – look there for a discussion on the various applications of integration!)

1. Administrivia and Announcements

2. HW comments

- Average: 87%, with a nice normal distribution centered around there.
- Common problems: None, really! People did very well on this HW set.

3. Limits - Two Definitions

So: when we write

$$\lim_{x \to b} f(x) = A,$$

what do we mean? Intuitively, we mean that whenever x is "close" to b, $f(x)$ is "close" to A – but how can we phrase this mathematically? We have two definitions, which are equivalent:

Definition 3.1. For a function f, we say that

$$\lim_{x \to b} f(x) = A$$

if and only if for every neighborhood N_A of A, there is a neighborhood N_b of b such that

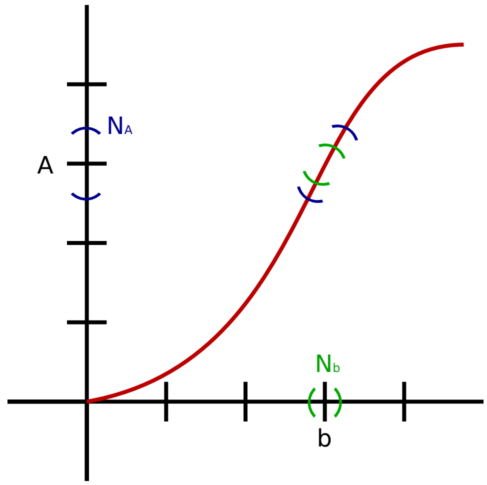
$$\forall x \in N_b, x \neq b, f(x) \in N_A.$$

(note that a **neighborhood** N of a point x is simply an open interval containing that point.)

Remark 3.2. What is this definition really saying? Well, it's basically saying that

$$\lim_{x \to b} f(x) = A$$

holds if points "close" to b (i.e. picking points in some neighborhood N_b) go to points "close" to A (i.e. points in our neighborhood N_A.) The picture below illustrates what's going on here:



There, however, is a "second" definition of the limit, using the $\epsilon - \delta$ notation – we write "second" because this definition is actually the same as the above definition, just with different notation and symbols.

Definition 3.3. For a function f, we say that

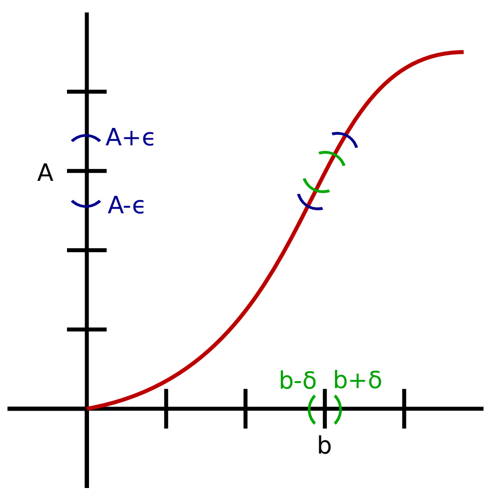
$$\lim_{x \to b} f(x) = A$$

if and only if for every $\epsilon > 0$, there is a $\delta > 0$ such that whenever a point x is within δ of b and is not b (i.e. $0 < |x - b| < \delta$), the point $f(x)$ is within ϵ of A (i.e. $|f(x) - A| < \epsilon$.)

Remark 3.4. What is this other definition really saying? Well, it's also basically saying that

$$\lim_{x \to b} f(x) = A$$

holds if points "close" to b (i.e. points within δ of b) go to points "close" to A (i.e. points within ϵ of A.) Look at the picture below of what's going on here:



This picture looks completely identical to our earlier picture! This is because these definitions are the exact same: all that the $\epsilon - \delta$ notation does is gives us a more concrete definition of what our "neighborhoods" actually are! For most of this class, we'll use the $\epsilon - \delta$ definition whenever we have to work something out from basic principles, but the neighborhood definition is probably the more intuitive one to think about when you're just trying to understand what's going on.

We should note, before moving on, the definition of continuity, as it relies so very heavily on limits:

Definition 3.5. We say that a function $f(x)$ is continuous at a point b iff

$$\lim_{x \to b} f(x) = f(b);$$

i.e. that $f(x)$ has a limit at b, and furthermore that limit is the value of the function itself at that point.

4. Limits - Tools

So: we, thankfully, have a number of tools and observations at hand to help us calculate limits. We review some of them here:

Proposition 4.1. *If the functions f, g have limits A, B such that*

$$\lim_{x \to c} f(x) = A, \lim_{x \to c} g(x) = B,$$

then we have the following properties:

- $$\lim_{x \to c} f(x) + g(x) = A + B,$$
- $$\lim_{x \to c} f(x) - g(x) = A - B,$$
- $$\lim_{x \to c} f(x) \cdot g(x) = A \cdot B.$$

As well, if $B \neq 0$, we have as well that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

Proposition 4.2. *All polynomials are continuous everywhere.*

Proposition 4.3. *All rational functions (that is, functions of the form $\frac{p(x)}{q(x)}$ where p, q are both polynomials) are continuous whenever their denominator $q(x)$ is nonzero.*

Proposition 4.4. $\sin(x)$ *and* $\cos(x)$ *are continuous everywhere.*

Proposition 4.5. *(Two-Policeman-Theorem / Squeeze Theorem) If I have three functions $f(x), g(x), h(x)$ and a neighborhood N_b of some point b such that*

(1)
$$f(x) \leq g(x) \leq h(x), \forall x \in N_b$$

(2)
$$\lim_{x \to b} f(x) = \lim_{x \to b} h(x),$$

then $g(x)$ has a limit as x approaches b, and furthermore

$$\lim_{x\to b} g(x) = \lim_{x\to b} h(x) = \lim_{x\to b} f(x).$$

5. Limits - Worked Examples

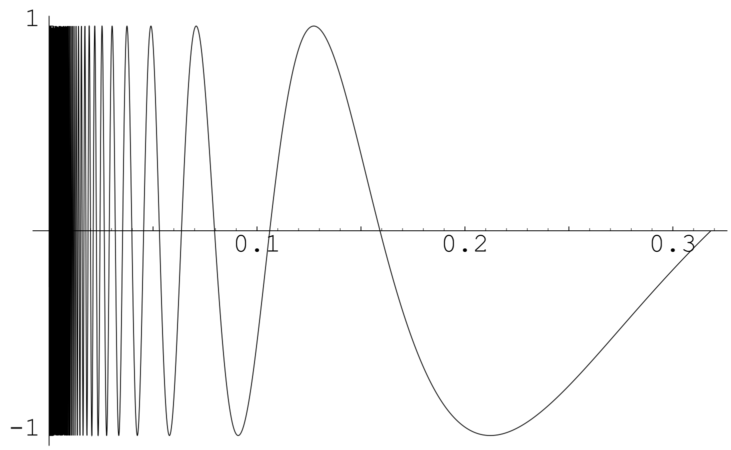
So: this, so far, is a lot of theory and not a lot of examples. Let's fix that.

Proposition 5.1. *We claim that*

$$\lim_{x\to 0} sin(1/x)$$

does not exist.

Proof. So: the first step in proving this is to remember what the graph of $\sin(1/x)$ actually looks like. A graph is attached below:



This certainly motivates, at least, the idea that this graph shouldn't have a limit at zero: no matter how close you get to the origin, $\sin(1/x)$ keeps taking on the values 1 and -1. We make this a bit more explicit below:

By basic trigonometry, we know that

$$\sin(x) = 1 \text{ for } x = \frac{4k\pi + 1}{2}$$

and

$$\sin(x) = -1 \text{ for } x = \frac{4k\pi + 3}{2},$$

□

for any integer k. So, this implies that

$$\sin(1/x) = 1 \text{ for } x = \frac{2}{4k\pi + 1}$$

and

$$\sin(1/x) = -1 \text{ for } x = \frac{2}{4k\pi + 3}.$$

So: we will prove that $\sin(1/x)$ has no limit at 0 by contradiction. Suppose not, that such a limit exists: call it A. Then, by definition, we know that for every $\epsilon > 0$

there is a $\delta > 0$ such that whenever $0 < |x| < \delta$, we have that $|\sin(1/x) - A| < \epsilon$. Thus, in particular, if $\epsilon = 1/2$, we know that there must be a δ such that $0 < |x| < \delta$, we have that $|\sin(1/x) - A| < 1/2$.

Pick n such that $\frac{2}{4n\pi+1}, \frac{2}{4n\pi+1} < \delta$; this is possible because large values of n make the denominators of these fractions very large, and thus the fractions themselves very small. Then, by the $\epsilon - \delta$ definition, we must have that

$$\left|\sin\left(\frac{1}{\frac{2}{4n\pi+1}}\right) - A\right| < 1/2$$

and

$$\left|\sin\left(\frac{1}{\frac{2}{4n\pi+3}}\right) - A\right| < 1/2;$$

but

$$\begin{aligned}
&\left|\sin\left(\frac{1}{\frac{2}{4n\pi+1}}\right) - A\right| < 1/2 \\
\Leftrightarrow &\left|\sin\left(\frac{4n\pi+1}{2}\right) - A\right| < 1/2 \\
\Leftrightarrow &|1 - A| < 1/2 \\
\Rightarrow &A > 1/2,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\sin\left(\frac{1}{\frac{2}{4n\pi+3}}\right) - A\right| < 1/2 \\
\Leftrightarrow &\left|\sin\left(\frac{4n\pi+3}{2}\right) - A\right| < 1/2 \\
\Leftrightarrow &|-1 - A| < 1/2 \\
\Rightarrow &A < -1/2.
\end{aligned}$$

This is clearly a contradiction, as A cannot be both greater than $1/2$ and less than $-1/2$. So our initial assumption must be false – i.e. $\sin(1/x)$ cannot have a limit at 0.

Proposition 5.2. *We claim that*

$$\lim_{x \to 0} x^2 \cdot \sin(1/x) \cdot \cos(e^{x^2} + 53 + 1/x^2) = 0.$$

Proof. Perhaps somewhat confusingly, this is a lot easier than the earlier proposition. Simply observe that

$$-1 \leq \sin(1/x) \leq 1,$$

for all $x \neq 0$, because sin is bounded between -1 and 1; similarly,

$$-1 \leq \cos(e^{x^2} + 53 + 1/x^2) \leq 1$$

by the same reasons.

So their product is bounded by the product of these bounds: i.e.

$$-1 \leq \sin(1/x)\cos(e^{x^2} + 53 + 1/x^2) \leq 1,$$

and thus (multiplying through by x^2) we have

$$-x^2 \leq x^2 \sin(1/x) \cos(e^{x^2} + 53 + 1/x^2) \leq x^2,$$

for all $x \neq 0$.

But we know that

$$\lim_{x \to 0} x^2 = \lim_{x \to 0} -x^2 = 0,$$

because polynomials are continuous: so applying the squeeze theorem/Two-Policeman theorem yields that

$$\lim_{x \to 0} x^2 \cdot \sin(1/x) \cdot \cos(e^{x^2} + 53 + 1/x^2) = \lim_{x \to 0} x^2 = \lim_{x \to 0} -x^2 = 0,$$

as claimed. □

Proposition 5.3. *We claim that*

$$\lim_{x \to 0} \frac{\sin(2x)}{x} = 2.$$

Proof. So: first recall from class/Apostol the limit result

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

From here, our claim is merely a trivial application of the double-angle formula:

$$\lim_{x \to 0} \frac{\sin(2x)}{x} = \lim_{x \to 0} \frac{2\sin(x)\cos(x)}{x} = \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \lim_{x \to 0} 2\cos(x) = 1 \cdot 2 = 2.$$

We note that splitting the limit here was OK, as both limits existed. □

Proposition 5.4. *We claim, furthermore, that*

$$\lim_{x \to 0} \frac{\sin(nx)}{x} = n.$$

Proof. We proceed by induction. We know, again from class/Apostol, that the base case

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

is true: so we proceed inductively.

Suppose that

$$\lim_{x \to 0} \frac{\sin(nx)}{x} = n.$$

We then seek to show that

$$\lim_{x \to 0} \frac{\sin((n+1)x)}{x} = n + 1.$$

To do this, we merely use the angle-addition formula:

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin((n+1)x)}{x} &= \lim_{x \to 0} \frac{\sin(nx)\cos(x) + \sin(x)\cos(nx)}{x} \\
&= \lim_{x \to 0} \frac{\sin(nx)\cos(x)}{x} + \lim_{x \to 0} \frac{\sin(x)\cos(nx)}{x} \\
&= \lim_{x \to 0} \frac{\sin(nx)}{x} \lim_{x \to 0} \cos(x) + \lim_{x \to 0} \frac{\sin(x)}{x} \lim_{x \to 0} \cos(nx) \\
&= n \cdot 1 + 1 \cdot 1 = n + 1,
\end{aligned}$$

where that last step came via using the inductive hypothesis. We note again that our various splits of the limits were all kosher, as in each case the limit existed. $\square$