MATH 1A, SECTION 1, WEEK 3 - RECITATION NOTES

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Abstract. These are the notes from Thursday, Oct. 15's recitation on integrals. Here, we review the step function definition of the integral, discuss the general definition of the integral, calculate the integral of several polynomials, and talk about the connections of the integral to the area between curves, volume, average values, and work. We close with an aside on polar coördinates, their definition, and an integral formula for functions of polar coördinates.

1. Administrivia and Announcements

2. HW comments

- Average: 70%, with a nice normal distribution centered around there.
- Common problems: Again, many students could stand to describe more of what they're doing in their proofs – there were several problems I saw worked with nary a syllable, which made them quite incomprehensible when (say) an arithmetic mistake or logical error would crop up. Also in the field of returning issues: many students forgot once more to use logical connectives ($\Leftrightarrow, \Rightarrow, \Leftarrow$) in their proofs and just had long chains of equations, which made their flow of logic fairly hard to understand.

  The only new issue that merited comment was people not quite understanding implication – specifically, the question that asked "Does $|1+3x| \leq 1$ imply $-2/3 \leq x$?" Pretty much everyone was able to manipulate the equations to see that

  $$(|1+3x| \leq 1) \Rightarrow (-2/3 \leq x \leq 0),$$

  but many people thought that this meant that $|1+3x| \leq 1$ **didn't** imply $-2/3 \leq x$, because it also implied that $0 \geq x$. But this is OK!

  This is actually pretty clear when you think about what implication means: when we say that $A \Rightarrow B$, we mean that whenever A is true, B must be true as well. So, if we say that $A \Rightarrow (B \text{ and } C)$, then we definitely have that A implies B – because whenever A is true, we have that B is true!

3. The Integral of a Step Function

So: we quickly review a few definitions:

Definition 3.1. A **partition** P of an interval $[a, b]$ is a finite ordered collection of real numbers $x_1 < x_2 < x_3 \ldots x_m$ such that $x_1 = a$ and $x_m = b$. This can be thought of as just a way to chop the interval $[a, b]$ into $m - 1$ different pieces.

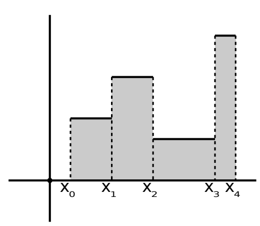
Definition 3.2. We call a function $s(x)$ on an interval $[a, b]$ a **step function** if we can find a partition $P = \{x_1, \dots x_m\}$ of $[a, b]$ and values $s_1 \dots s_{m-1}$ such that

$$s(x) = s_i \text{ on } (x_i, x_{i+1})$$

Definition 3.3. Take a step function s on an interval $[a, b]$ that takes the values s_k on a partition $P = \{x_1 \dots x_m\}$ of $[a, b]$. We then define the **integral** of s on $[a, b]$ as follows:

$$\int_a^b s(x)dx = \sum_{k=1}^{m-1} s_k \cdot (x_{k+1} - x_k).$$

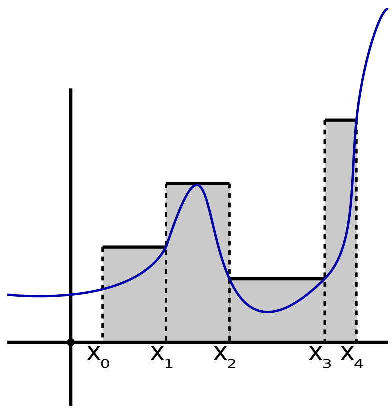
Why do we use this definition? Well: one way of interpreting the integral of a function is that it is a way of measuring the "area" underneath the curve of the function. If we apply this idea to a step function, we would naturally conclude that the integral of such a step function would just be the sum of the "rectangles" bounded on top by such a step function – but the area of each individual rectangle is just their height (i.e. $s(x)$'s value over that interval) times their width (i.e. the length of the interval. In other words: precisely the definition we have chosen.



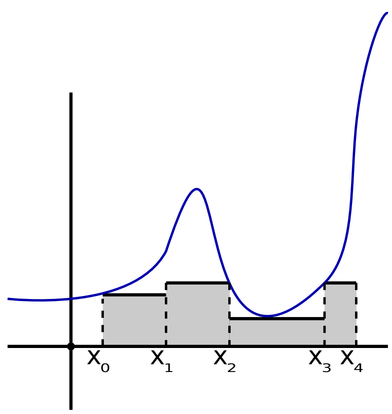
4. The Integral for More General Functions - Definition

So: we often want to find the area under the curve for many functions, not just step functions. How can we do this?

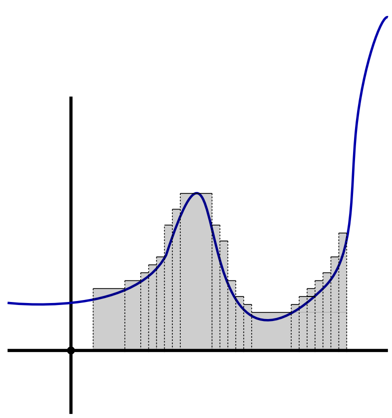
Well, one method might be to try to approximate our function by taking the area of a collection of rectangles that bound our function from above:



Alternately, we could try taking the area of collections of rectangles that bound our function from below:



In either situation, if we pick sufficiently small rectangles, we should, intuitively, get a number that is pretty close to the total area:



This intuition should hopefully motivate our definition of the integral for a general function, which we state here.

Definition 4.1. A function f has an integral over the interval $[a, b]$ iff

$$\inf\left\{\int_a^b s(x) : s(x) \leq f(x), \forall x \in [a,b]\right\} = \sup\left\{\int_a^b t(x) : t(x) \geq f(x), \forall x \in [a,b]\right\}.$$

In other words, a function f has an integral if and only if trying to approximate its area from above yields the same answer as trying to approximate it from below. If this holds, then we define the integral of f to be

$$\int_a^b f(x)dx = \inf\left\{\int_a^b s(x) : s(x) \leq f(x)\right\} = \sup\left\{\int_a^b t(x) : t(x) \geq f(x)\right\}.$$

5. The Integral for More General Functions - an Example

So: how do we actually calculate such a ponderous thing? We illustrate with an example calculation, from Apostol:

Proposition 5.1.

$$\int_0^b x^p dx = \frac{b^{p+1}}{p+1}.$$

Proof. So: Take the following partition

$$0 < \frac{b}{n} < \frac{2b}{n} < \frac{3b}{n} < \ldots < \frac{(n-1)b}{n} < \frac{nb}{n} = b$$

of $[0, b]$.

Define the step function $s(x)$ on this partition as follows:

$$s(x) = \left(\frac{kb}{n}\right)^p \text{ on } \left[\frac{kb}{n}, \frac{(k+1)b}{n}\right].$$

It should be clear from our definition that $s(x) \leq x^p$ for every $x \in [0, b]$.

Similarly, define the step function $t(x)$ on this partition as follows:

$$t(x) = \left(\frac{(k+1)b}{n}\right)^p \text{ on } \left[\frac{kb}{n}, \frac{(k+1)b}{n}\right].$$

It should also be clear from our definition that $t(x) \geq x^p$ for every $x \in [0, b]$.

So: what are the integrals of these two step functions?

Well, by definition,

$$\begin{aligned}
\int_0^b s(x)dx &= \sum_0^{k=n-1} \left(\frac{(k+1)b}{n} - \frac{kb}{n}\right) \cdot \left(\frac{kb}{n}\right)^p \\
&= \sum_0^{k=n-1} \left(\frac{b}{n}\right) \cdot \left(\frac{kb}{n}\right)^p \\
&= \left(\frac{b^{p+1}}{n^{p+1}}\right) \cdot \sum_0^{k=n-1} \cdot (k)^p ,
\end{aligned}$$

and

$$\begin{aligned}\int_0^b t(x)dx &= \sum_0^{k=n-1} \left(\frac{(k+1)b}{n} - \frac{kb}{n}\right) \cdot \left(\frac{(k+1)b}{n}\right)^p \\ &= \sum_0^{k=n-1} \left(\frac{b}{n}\right) \cdot \left(\frac{(k+1)b}{n}\right)^p \\ &= \left(\frac{b^{p+1}}{n^{p+1}}\right) \cdot \sum_0^{k=n-1} \cdot (k+1)^p \\ &= \left(\frac{b^{p+1}}{n^{p+1}}\right) \cdot \sum_0^{k=n} \cdot (k)^p .\end{aligned}$$

Taking the difference, we then have that

$$\begin{aligned}\int_0^b t(x)dx - \int_0^b s(x)dx &= \left(\frac{b^{p+1}}{n^{p+1}}\right) \cdot \sum_0^{k=n} \cdot (k)^p - \left(\frac{b^{p+1}}{n^{p+1}}\right) \cdot \sum_0^{k=n-1} \cdot (k)^p \\ &= \left(\frac{b^{p+1}}{n^{p+1}}\right) \cdot \left(\sum_0^{k=n} \cdot (k)^p - \sum_0^{k=n-1} \cdot (k)^p\right) \\ &= \left(\frac{b^{p+1}}{n^{p+1}}\right) \cdot n^p \\ &= \frac{b^{p+1}}{n},\end{aligned}$$

which goes to 0 as n gets increasingly large. As a result, we have that

$$\inf\left\{\int_a^b s(x) : s(x) \le x^p\right\} = \sup\left\{\int_a^b t(x) : t(x) \ge x^p\right\},$$

and thus that the integral exists!

To see what it actually is: remember from week 2's HW that we showed that

$$\sum_0^{k=n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_0^{k=n} k^p;$$

if we multiply all sides of this equation by $\frac{b^{p+1}}{n^{p+1}}$, we then have

$$\left(\frac{b^{p+1}}{n^{p+1}}\right) \cdot \sum_0^{k=n-1} k^p < \left(\frac{b^{p+1}}{n^{p+1}}\right) \cdot \frac{n^{p+1}}{p+1} = \frac{b^{p+1}}{p+1} < \left(\frac{b^{p+1}}{n^{p+1}}\right) \cdot \sum_0^{k=n} k^p.$$

The left and right hand sides of this equation should look rather familiar to you – they're in fact the integrals of $s(x)$ and $t(x)$ that we just calculated above! So, substituting, we then have

$$\int_0^b s(x)dx < \frac{b^{p+1}}{p+1} < \int_0^b t(x)dx.$$

But as we noticed above, the distance between these two integrals goes to 0 as n gets very large. So, as a result, if these two integrals are going to the same place and $\frac{b^{p+1}}{p+1}$ always lies between them, we then have that they must both be converging to $\frac{b^{p+1}}{p+1}$ – and thus that $\frac{b^{p+1}}{p+1}$ is the integral of x^p from 0 to b. $\square$

So: with this in mind, we can integrate any polynomial over any interval, if we use the linearity and additivity of the integral. We do an illustrative example below:

Example 5.2. The integral of

$$\int_{-2}^{2} x^2 dx$$

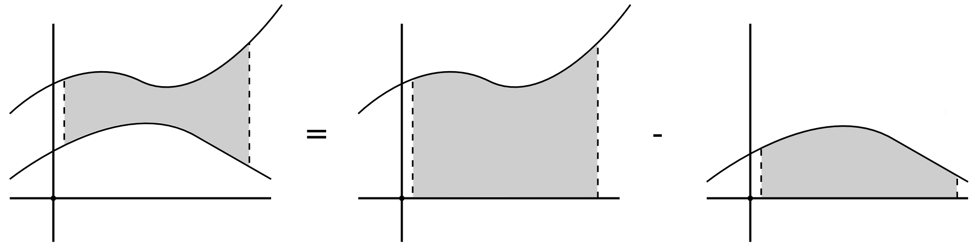
can be calculated thusly:

$$\begin{aligned}
\int_{-2}^{2} x^2 dx &= \int_{-2}^{0} x^2 dx + \int_{0}^{2} x^2 dx && \text{by additivity of the integral}\\
&= \int_{0}^{2} (x-2)^2 dx + \int_{0}^{2} x^2 dx && \text{invariance under translation}\\
&= \int_{0}^{2} (x^2-4x+4) dx + \int_{0}^{2} x^2 dx && \text{algebra}\\
&= \int_{0}^{2} x^2 dx - 4\int_{0}^{2} x dx + 4\int_{0}^{2} dx + \int_{0}^{2} x^2 dx && \text{linearity of the integral}\\
&= \frac{2^3}{3} - 4\cdot\frac{2^2}{2} + 4\cdot\frac{2^1}{1} + \frac{2^3}{3} && \text{int. of } x^p\\
&= \frac{16}{3}
\end{aligned}$$

6. Applications of the Integral

So, it's worth discussing briefly several things we can do with the integral:

(1) Finding the area between curves: If we want to find the area between two curves graphed by functions a and b, we can simply take the difference of their integrals: i.e. examine $\int_a^b f(x) - g(x)$. The reasoning for this is perhaps made clearest by looking at a picture:



(2) Integrals of trigonometric functions:

$$\int_a^b \sin(x)dx = -\cos(b) + \cos(a),$$
$$\int_a^b \cos(x)dx = \sin(b) - \sin(a).$$

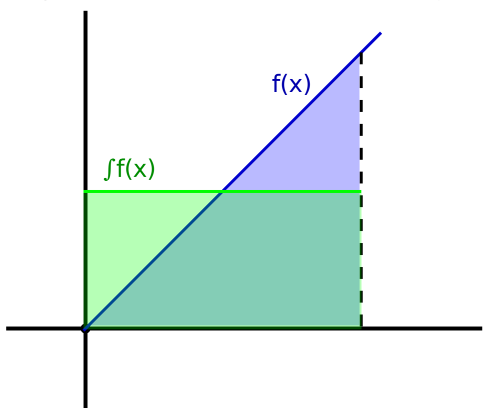
The derivations are done in Apostol; it's worth looking at them if you aren't comfortable with this material yet.

(3) Volume: if $f(x)$ represents a function that gives you the cross-sectional area at height x of some shape, then $\int_a^b f(x)$ will give you the volume of that shape that lies within those height bounds. This should be fairly intuitive, as volume is just area multiplied by height for simple shapes.

(4) Average value: For a function f, the **average value** of f on some interval $[a, b]$ is just

$$\frac{1}{b-a} \cdot \int_a^b f(x)dx.$$

Again, the reasoning for this can be best illustrated by a picture:



In the graph above, we have the graph of $f(x)$ placed over a constant function with value $1/(b-a) \cdot \int_a^b f(x)$. By definition, these two curves have the same area; so (logically) we would expect one of them to be larger than the other about "half" of the time – i.e. that the constant value $1/(b-a) \cdot \int_a^b f(x)$ is roughly the "average value" of $f(x)$.

(5) Work: If we have a function f that represents the force applied to an object as it moves, then the total **work** done in moving that object from point a to point b is just

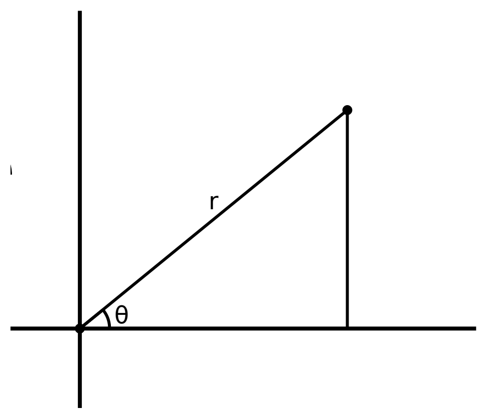
$$\int_a^b f(x)dx.$$

7. Polar Coördinates

So: first, what do we mean by the word "coördinates" in mathematics? For the purposes of this class, a "coördinate system" is a way to identify the location of points in some given space – most often, this space will be $\mathbb{R}^2$ or $\mathbb{R}^3$. The coördinate system you're all familiar with already is the Cartesian coördinate system, where we identify the location of a point in $\mathbb{R}^n$ by giving its distance from the n different axes in $\mathbb{R}^n$. For example, in $\mathbb{R}^2$, we identify a point by giving its distance from the x an y axes, and describe that point with this pair of distances (x, y).

However, there are other coördinate systems! In particular, in $\mathbb{R}^2$, we have the **polar coordinate system**, in which we identify a point in $\mathbb{R}^2$ by giving two pieces of information:

- its distance from the origin, and
- the angle formed from starting at the positive x-axis and rotating counterclockwise until we get to the line containing this point.



The picture above illustrates a point in $\mathbb{R}^2$ with polar coördinates (r, θ).

Given a point (r, θ) in polar coördinates, we can describe this same point in Cartesian coördinates as

$$x = r\cos(\theta),$$
$$y = r\sin(\theta);$$

conversely, given a point (x, y) in Cartesian coördinates, we can describe it in polar coördinates via the equations

$$r = \sqrt{x^2 + y^2},$$
$$\theta = \arctan(y/x).$$

Basic trigonometry will verify these properties.

So: we can define functions in polar coördinates as well – these are typically defined as functions that take in given angles and spit out various radii. For example, the function $r(\theta) = 1$ will graph a circle, while the function $r(\theta) = \cos(2\theta)$ will graph a four-petaled rose.

We didn't have time to explain this in recitation, but we can talk about the integrals of polar functions, just as we talked about the integrals of normal functions – explicitly, if we want to calculate the area between a polar curve and the origin, between angles a and b, we simply need to calculate

$$\frac{1}{2} \cdot \int_a^b (f(\theta))^2 d\theta.$$

A proof of this property is in Apostol; I'm more than happy to go over it with any curious students.