# MATH 1A, SECTION 1, WEEK 11 - FINAL REVIEW NOTES

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Abstract. These are the notes from Monday, Dec. 7th's recitation, which aims to review the entire second half of the course and work several examples along the way.

## 1. What You Have (hopefully) Learned

1.1. **Continuous Functions and their Properties.** Continuous functions have lots of useful properties! Here are a few you should know:

- **The Definition of Continuity**. Hopefully, this is second nature by now.
- **Continuity of Integrals**. Integrals are continuous.
- **Intermediate Value Theorem**. Basically: if a function $f$ is continuous on a closed interval $[a, b]$, it hits every value between $f(a)$ and $f(b)$ on $(a, b)$.
- **Boundedness Theorem for Continuous Functions**. Continuous functions are bounded on closed intervals.
- **Extreme Value Theorem**. If $f$ is a continuous function on a closed interval $[a, b]$, then there is some $c \in [a, b]$ such that $f(c) = \sup_{[a,b]}(f)$.
- **Mean Value Theorem(integral)**. If $f$ is continuous on a closed interval $[a, b]$, then there is some $c \in [a, b]$ such that $\int_a^b f = (b - a) \cdot f(c)$.
- **Mean Value Theorem(derivative)** If $f$ is continuous on a closed interval $[a, b]$ and also differentiable on $(a, b)$, then there is a point $c \in (a, b)$ so that $f(b) - f(a) = (b - a) \cdot f'(c)$.

1.2. **The Derivative.** We've calculated a number of derivatives in the last six weeks, and used them in a variety of ways. A quick list of the techniques we've developed so far follows below:

- **The Definition of the Derivative**. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Always handy for when you're trying to calculate a derivative of something you can't understand any other way. Also useful for proving that a derivative doesn't exist, or finding the derivative of a piecewise-defined function.
- **Product Rule**. $(fg)' = f'g + fg'$. Should be second nature by now.
- **Quotient Rule**. $\left(\frac{f}{g}\right)' = \frac{f'g-g'f}{g^2}$. A special case of the product rule.
- **Chain Rule**. $(f(g(x))' = f'(g(x)) \cdot g'(x)$. This also is something you should be able to do in your sleep.
- **Logarithmic Differentiation**. $f'(x) = (\log(f(x))' \cdot f(x)$. Useful for things like $x^x$.
- **Special Derivatives** Know the derivatives of the inverse trig functions $\frac{1}{1+u^2}, \frac{1}{\sqrt{1-x^2}}, -\frac{1}{\sqrt{1-x^2}}$, and know how to differentiate $\log(x)$ and $e^x$.
- **Optimization with Derivatives**. Basically: differentiable functions can only have maximums at critical points (places where $f' = 0$) or at the

endpoints of the interval. Examining the sign of the second derivative will tell you whether a given point is a local minimum or a local maximum.

- **Graphing and Derivatives**. Remember: the first derivative tells you whether the function is decreasing or increasing; the second derivative tells you whether the first derivative is decreasing or increasing (i.e. the curvature of the graph). Be able to identify regions where the graph is concave up or down, find inflection points, etc.

1.3. **The Integral.** While we introduced the integral in the first half of our course, the second half is where we really developed some calculational techniques. We list some of these methods below:

- **The First and Second Fundamental Theorems of Calculus**. The first says that if $A(x) = \int_c^x f(t)dt$, then $A'(x) = f(x)$; the second says that if $F$ is a primitive of $f$, then $F(x) = F(c) + \int_c^x f(t)dt$ (up to the conditions on the functions $f, A$, and $F$.)
- **Integration by Parts**. $\int udv = uv - \int vdu$. Basically the product rule in reverse.
- **Integration by Substitution.** $\int f(u(x))u'(x)dx = \int f(x)dx$. Basically the chain rule in reverse.
- **Special Integrals**. Know the integrals that give you the inverse trig functions $\frac{1}{1+u^2}, \frac{1}{\sqrt{1-x^2}}, -\frac{1}{\sqrt{1-x^2}}$, and know how to integrate $\log(x)$ and $e^x$.
- **Partial Fractions**. Remember: before you can apply the method of partial fractions, you have to make sure that the numerator has smaller degree than the denominator (via long division)! Look at examples if you're still shaky on the methods.

1.4. **The Logarithm and the Exponential.** We've used these functions extensively throughout this course; by this point, you should be quite used to to calculating integrals and derivatives with them.

- **Definitions.** Know them.
- **Additive and Multiplicative Properties.** I.e. $\log(xy) = \log(x) + \log(y), e^{x+y} = e^x e^y$. Know these! If you get confused about how the properties work, do a quick test case with $x = 1, y = 1$; this will usually tell you if you're right or wrong.

1.5. **The Taylor Polynomial.** The last major subject that will appear on your final, Taylor polynomials are relatively simple things (all they involve is derivatives and sums!) that nonetheless seem to trip a lot of people up. Be capable of calculating a Taylor polynomial of a given degree for a function, and of bounding a given-degree error function.

- **Definition.** $T_n(f)(x) = \sum_{k=0}^n \frac{f^{(k)}(c)}{k!} \cdot (x-c)^k$. Know it.
- **Calculations**. More importantly, be able to calculate the Taylor polynomial of a given function. All you need to be able to do this is the ability to take derivatives and plug them into the above equation, so this shouldn't be too bad.
- **Error Function - Definition**. $E_n(f)(x) = f(x) - T_n(f)(x)$. Know it.
- **Error Function - Massively Useful Theorem**. If $f$ has its $n+1$-th derivative bounded below by $m$ and above by $M$ on some interval $[a, b]$,

then $E_n(f)$ is also bounded there! Explicitly,

$$m \cdot \frac{(x-a)^{n+1}}{(n+1)!} \leq E_n(f)(x) \leq M \cdot \frac{(x-a)^{n+1}}{(n+1)!}, \qquad x > c,$$
$$m \cdot \frac{(a-x)^{n+1}}{(n+1)!} \leq (-1)^{n+1} E_n(f)(x) \leq M \cdot \frac{(a-x)^{n+1}}{(n+1)!}, \qquad x < c,$$

where $c$ is the constant we took our Taylor polynomial around. This is probably the longest thing you have to memorize for this test; but I would strongly recommend doing so.

## 2. Review Questions

I've written up five questions below that should hopefully test your understanding of the above material; if you can do these six problems, you should do quite well on the final!

## 3. Exercise 1

**Question 3.1.** *Suppose that $f$ is a function such that $f''(x) = -3$, and $f'(1) = f(1) = 0$. Find $f$.*

*Proof.* So: by the second fundamental theorem of calculus, we know that (because $f'$ is a primitive of $f''$)

$$\begin{aligned} f'(x) &= f'(1) + \int_1^x f''(t)dt \\ &= 0 + \int_1^x -3dt \\ &= -3x + 3. \end{aligned}$$

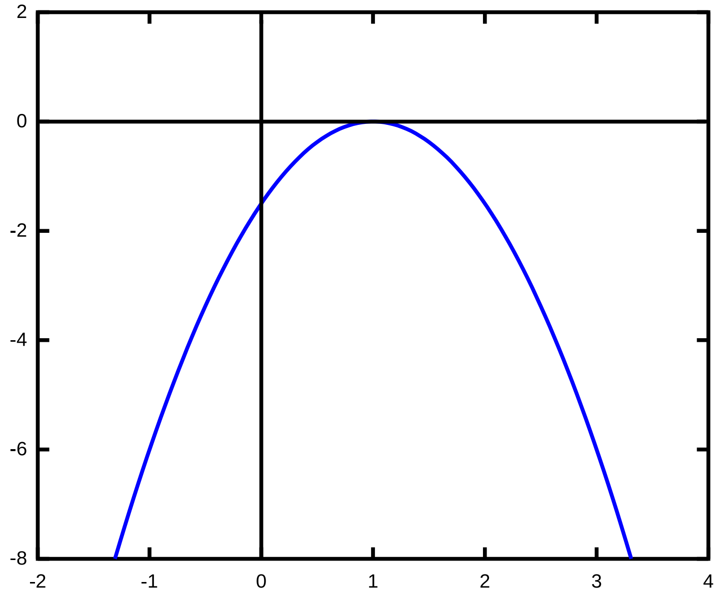
Applying the second fundamental theorem again (because $f$ is a primitive of $f'$), we have that

$$\begin{aligned} f(x) &= f(1) + \int_1^x f'(t)dt \\ &= 0 + \int_1^x -3x + 3dt \\ &= -\frac{3x^2}{2} + 3x - \frac{3}{2} \\ &= -\frac{3}{2}(x-1)^2. \end{aligned}$$

□

**Question 3.2.** *Without using the above results, try to graph $f$.*

*Proof.* So: because the second derivative of $f$ is always $< 0$, we know that the first derivative of $f$ is always decreasing and also continuous (by the continuity of the integral + realizing that $(f')' = f''$). Because $f'(1) = 0$, we then know that $f'(x)$ is only equal to 0 at 1 (as it's always decreasing), and thus that $f$ has its only critical point at 0. Because $f'' < 0$, we know that this point is a maximum and the graph is concave-down everywhere; applying that $f(1) = 0$ then tells us that the graph is pretty much of the form drawn below:




$\square$

## 4. Exercise 2

**Question 4.1.** *Take the derivative of* $x^{x^x}$.

*Proof.* We use logarithmic differentiation, and calculate:

$$\begin{aligned}
\left(x^{x^x}\right)' &= \left(\log\left(x^{x^x}\right)\right)' \cdot x^{x^x} \\
&= \left(x^x \cdot \log(x)\right)' \cdot x^{x^x} \\
&= \left((x^x)' \cdot \log(x) + x^x \cdot \frac{1}{x}\right) \cdot x^{x^x} \\
&= \left(\left(e^{x\log(x)}\right)' \cdot \log(x) + x^x \cdot \frac{1}{x}\right) \cdot x^{x^x} \\
&= \left(\left(x \cdot \frac{1}{x} + \log(x)\right) \cdot e^{x\log(x)} \cdot \log(x) + x^x \cdot \frac{1}{x}\right) \cdot x^{x^x} \\
&= \left((1+\log(x)) \cdot x^x \cdot \log(x) + x^{x-1}\right) \cdot x^{x^x}.
\end{aligned}$$

$\square$

## 5. Exercise 3

**Question 5.1.** *Show that the function*

$$f(x) = \frac{x(x-1)\cdot\sin(x)}{\cos^2(x)} + \frac{(x-2)\cdot\log|x+1|}{(x+1)^3} + (x-1)(x-2)\cdot e^{x^x}\cdot\cos(x)$$

*has at least two zeroes in* $(0, 2)$.

*Proof.* So: notice that

$$\begin{aligned}
f(0) &= \frac{0(0-1)\cdot\sin(0)}{\cos^2(0)} + \frac{(0-2)\cdot\log|0+1|}{(0+1)^3} + (0-1)(0-2)\cdot e^{0^0}\cdot\cos(0) \\
&= 0 + 0 + (-1)(-2)e^0\cdot 1 \\
&= 2, \\
f(1) &= \frac{1(1-1)\cdot\sin(1)}{\cos^2(1)} + \frac{(1-2)\cdot\log|1+1|}{(1+1)^3} + (1-1)(1-2)\cdot e^{1^1}\cdot\cos(1) \\
&= 0 + \frac{-1\log(2)}{8} + 0 \\
&= -\frac{\log(2)}{8}, \\
f(2) &= \frac{2(2-1)\cdot\sin(2)}{\cos^2(2)} + \frac{(2-2)\cdot\log|2+1|}{(2+1)^3} + (2-1)(2-2)\cdot e^{2^2}\cdot\cos(2) \\
&= \frac{2\sin(2)}{\cos^2(2)} + 0 + 0 \\
&= \frac{2\sin(2)}{\cos^2(2)}.
\end{aligned}$$

Specifically: $f(0) > 0, f(1) < 0, f(2) > 0$.

So: recall the intermediate value theorem, which states that if $f$ is a continuous function on some interval $[a, b]$ then it adopts every value between $f(a)$ and $f(b)$ on the open interval $(a, b)$. In our situation, we have that $f$ is continuous on the interval $[0, 1]$, because it's a product and sum of functions which are continuous on this interval; so it takes on every value between $f(0)$ and $f(1)$ on the open interval $(0, 1)$. Specifically, because $f(0) > 0$ and $f(1) < 0$, we know that there is some $a \in (0, 1)$ such that $f(a) = 0$.

Similarly, because $f$ is continuous on $[1, 2]$ and $f(1) < 0, f(2) > 0$, we know that there is some $b$ in $(1, 2)$ such that $f(b) = 0$. This gives us two distinct zeroes for $f$, which is what we wanted to show. □

**Question 5.2.** *Conclude that there is some point* $c \in (0, 2)$ *such that* $f'(c) = 0$.

*Proof.* So: recall the mean value theorem for derivatives, which states that any continuous function on $[a, b]$ that's differentiable on $(a, b)$ has a point $c \in (a, b)$ such that $(b - a)f'(c) = f(b) - f(a)$.

Explicitly, if we let $a, b$ be from the earlier part of our question, we know that $f$ is continuous and differentiable on $[a, b]$, and thus that there is some $c \in (a, b) \subset (0, 2)$ such that

$$\begin{aligned}(b-a)f'(c) &= f(b)-f(a)\\ \Rightarrow f'(c) &= \frac{f(b)-f(a)}{b-a}\\ \Rightarrow f'(c) &= \frac{0-0}{b-a}\\ \Rightarrow f'(c) &= 0.\end{aligned}$$

□

## 6. Exercise 4

**Question 6.1.** *Calculate*

$$\int \frac{x^4-x^3+3x^2-2x+1}{x^3-x^2+x-1}dx$$

*Proof.* So: we use the method of partial fractions. To reduce the degree of the numerator, we use long division to see that

$$\begin{array}{r|l} & x \\ \hline x^3-x^2+x-1 & x^4-x^3+3x^2-2x+1 \\ & -x^4+x^3-x^2+x \\ \hline & 2x^2-x \end{array}$$

and thus that

$$\int \frac{x^4-x^3+3x^2-2x+1}{x^3-x^2+x-1} = \int x + \frac{2x^2-x+1}{x^3-x^2+x-1}dx.$$

We can factor the denominator as $(x-1)(x^2+1)$, which tells us that we're looking for constants $A, B, C$ such that

$$\frac{2x^2-x+1}{x^3-x^2+x-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

This is equivalent to the three equations

$$\begin{aligned}2x^2 &= Ax^2+Bx^2\\ -x &= -Bx+Cx\\ 1 &= A-C\end{aligned}$$

which force $A=B=1, C=0$, and tell us that

$$\begin{aligned}\int x + \frac{2x^2-x+1}{x^3-x^2+x-1}dx &= \int x + \frac{1}{x-1} + \frac{x}{x^2+1}dx\\ &= \frac{x^2}{2} + \log|x-1| + \frac{1}{2}\log|x^2+1| + C.\end{aligned}$$

□

## 7. Exercise 5

**Question 7.1.** *Prove that the $n$-th Taylor polynomial for $e^{x/2}$ is*

$$T_n(e^{x/2}) = \sum_{k=0}^{n} \frac{1}{2^k k!} x^k.$$

*Proof.* So: recall the general formula for Taylor series, which says that

$$T_n(e^{x/2}) = \sum_{k=0}^{n} \frac{\frac{d^k}{dx^k}\left(e^{x/2}\right)\Big|_0}{k!} x^k.$$

This tells us that if we can find a general form for the derivative of $e^{x/2}$, we can just evaluate it at zero and plug it into the above formula to find $e^{x/2}$'s $n$-th Taylor polynomial. So, let's do just that!

To find a pattern, we first calculate a few derivatives:

$$\begin{aligned}
\frac{d^0}{dx^0}\left(e^{x/2}\right) &= e^{x/2} \\
\frac{d^1}{dx^1}\left(e^{x/2}\right) &= \frac{e^{x/2}}{2} \\
\frac{d^2}{dx^2}\left(e^{x/2}\right) &= \left(\frac{e^{x/2}}{2}\right)' = \frac{e^{x/2}}{4} \\
\frac{d^3}{dx^3}\left(e^{x/2}\right) &= \left(\frac{e^{x/2}}{4}\right)' = \frac{e^{x/2}}{8} \\
\frac{d^4}{dx^4}\left(e^{x/2}\right) &= \left(\frac{e^{x/2}}{8}\right)' = \frac{e^{x/2}}{16}
\end{aligned}$$

The pattern above seems to be that $\frac{d^k}{dx^k}\left(e^{x/2}\right) = \frac{e^{x/2}}{2^k}$. We claim that this is true, and prove it by induction:

Base case: trivial, as $\frac{d^0}{dx^0}\left(e^{x/2}\right) = e^{x/2} = \frac{e^{x/2}}{2^0}$.

Inductive step: if $\frac{d^k}{dx^k}\left(e^{x/2}\right) = \frac{e^{2x}}{2^k}$, then

$$\frac{d^{k+1}}{dx^{k+1}}\left(e^{x/2}\right) = \left(\frac{e^{2x}}{2^k}\right)' = \frac{1}{2}\frac{e^{2x}}{2^k} = \frac{e^{2x}}{2^{k+1}}.$$

This proves that the $k$-th derivative of $e^{x/2}$ is $\frac{e^{2x}}{2^k}$. So, if we plug in zero, we get that the $k$-th derivative of $e^{x/2}$ at zero is just $\frac{1}{2^k}$; plugging this into the formula for Taylor series gives us that

$$T_n(e^{x/2}) = \sum_{k=0}^{n} \frac{1}{2^k k!} x^k,$$

as we claimed. $\square$

**Question 7.2.** *Use the third-order Taylor series approximation calculated above to bound the integral*

$$\int_{1/2}^{1} \frac{e^{x/2}}{x} dx.$$

*Proof.* So: from the above, we know that we can write

$$e^{x/2} = 1 + \frac{x}{2} + \frac{x^2}{4 \cdot 2} + \frac{x^3}{8 \cdot 3!} + E_3(e^{x/2}),$$

and thus we can rewrite our integral as

$$\int_{1/2}^{1} \frac{1 + \frac{x}{2} + \frac{x^2}{4\cdot 2} + \frac{x^3}{8\cdot 3!} + E_3(e^{x/2})}{x} dx$$

So, we now want to bound $E(e^{x/2})$! To do this, we use the bounding property we discussed earlier in this paper, which we restate here: If $f$ has its $n+1$-th derivative bounded below by $m$ and above by $M$ on some interval $[a, b]$, then

$$m \cdot \frac{(x-a)^{n+1}}{(n+1)!} \leq E_n(f)(x) \leq M \cdot \frac{(x-a)^{n+1}}{(n+1)!}, \qquad x > a.$$

In our specific case, $f(x) = e^{x/2}$ and $n = 3$. In this case, the $3 + 1 = 4$-th derivative of $f$ is $\frac{e^{x/2}}{16}$, as we showed earlier. We can see that this is bounded on $[1/2, 1]$ below by 0 (as $e^{x/2}/16$ is always positive,) and above by $e^{1/2}/16 < 2/16 = 1/8$ (because $e^{x/2}/16$ is increasing, and thus takes its maximal value at $x = 1$.) Thus, we know that

$$0 \leq E_n(f)(x) \leq \frac{1}{8} \cdot \frac{x^4}{4!} \text{ on } [1/2, 1].$$

This allows us to bound the integral $\int_{1/2}^{1} \frac{e^{x/2}}{x} dx$ from below, as

$$\begin{aligned}
\int_{1/2}^{1} \frac{1 + \frac{x}{2} + \frac{x^2}{4\cdot 2} + \frac{x^3}{8\cdot 3!} + E_3(e^{x/2})}{x} dx &\geq \int_{1/2}^{1} \frac{1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48}}{x} dx \\
&= \int_{1/2}^{1} \frac{1}{x} + \frac{1}{2} + \frac{x}{8} + \frac{x^2}{48} dx \\
&= \left( \log|x| + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{144} \right) \Bigg|_{1/2}^{1} \\
&= \frac{1}{2} + \frac{1}{16} + \frac{1}{144} + \log(1) - \frac{1}{4} - \frac{1}{64} - \frac{1}{1152} - \log 1/2 \\
&= \frac{349}{1152} + \log 2.
\end{aligned}$$

Similarly, we can bound this integral from above using the observation that $E_3(e^{x/2}) \leq \frac{x^4}{192}$:

$$\begin{aligned}
\int_{1/2}^{1} \frac{1 + \frac{x}{2} + \frac{x^2}{4 \cdot 2} + \frac{x^3}{8 \cdot 3!} + E_3(e^{x/2})}{x} dx &\leq \int_{1/2}^{1} \frac{1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{192}}{x} dx \\
&= \int_{1/2}^{1} \frac{1}{x} + \frac{1}{2} + \frac{x}{8} + \frac{x^2}{48} + \frac{x^3}{192} dx \\
&= \left( \log|x| + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{144} + \frac{x^4}{768} \right) \Bigg|_{1/2}^{1} \\
&= \frac{1}{2} + \frac{1}{16} + \frac{1}{144} + \frac{1}{768} + \log(1) - \frac{1}{4} - \frac{1}{64} - \frac{1}{1152} - \frac{1}{12288} \log 1/2 \\
&= \frac{349}{1152} + \log 2 + \frac{15}{12288}
\end{aligned}$$

So the answer lies between $\frac{349}{1152} + \log 2$ and $\frac{349}{1152} + \log 2 + \frac{15}{12288}$.

$\square$