MATH 1A, SECTION 1, WEEK 10 - RECITATION NOTES

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Abstract. These are the notes from Thursday, Dec. 3th's recitation, on Taylor polynomials. Here, we define the Taylor polynomial, calculate several examples, and discuss how these approximations differ from the functions they are modeling.

1. Homework comments

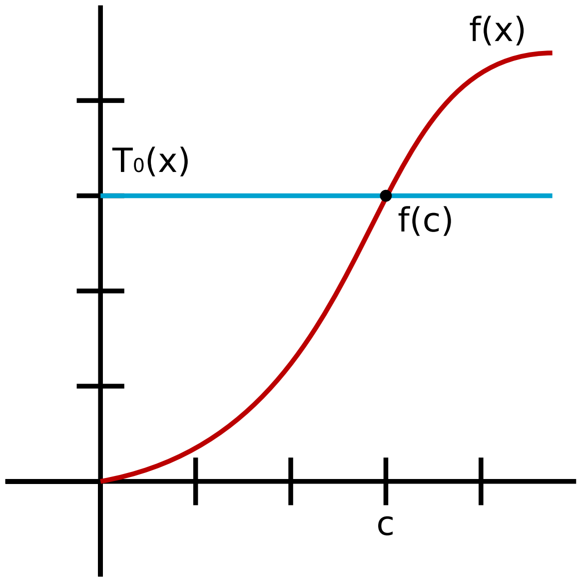
- Homework 7 average: 87%. Homework 8 average: 89%.
- Comments: People did well! Comments are accordingly minimal:
  - Before using the methods of partial fractions, make sure that the degree of the polynomial in the numerator is lower than the degree of the polynomial in the denominator! This tripped a number of people up.
  - (as always:) Words! Use them.

2. Taylor Series - Motivation

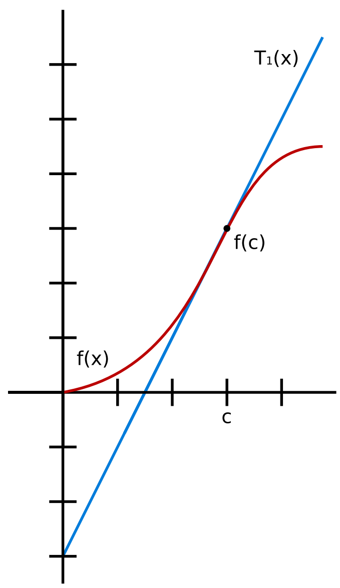
As you may have noticed in this course, there are many functions in mathematics that are "hard to work with" – i.e. their indefinite integrals may be messy to compute, or perhaps even impossible with the methods we've described thus far in this course. (Examples of such functions are $\sin(x)/x$ and $e^{-t^2/2}$.) Yet, we will often want to work with and study such functions – the first, $\sin(x)/x$, comes up in signal processing, and the second, $e^{-t^2/2}$, is the normal distribution (and thus shows up everywhere.)

How can we study such functions? One method, which we outline in this recitation, is the study of Taylor's method for approximating these functions by polynomials. The motivation for this is the following situation: Suppose that you have some function $f(x)$, and you want to create a polynomial that will "look like" $f(x)$. Well, maybe it's going to be impossible to make a polynomial that looks like $f(x)$ everywhere – for example, no polynomial approximation to $\sin(x)$ will ever be very good everywhere, because $\sin(x)$ is an oscillating bounded function and any non-constant polynomial is unbounded on $\mathbb{R}$. So we'll attempt the easier task of trying to approximate $f(x)$ just around some point c.

So: how can we do this? Well, suppose you're trying to approximate $f(x)$ by some polynomial T_0 of degree 0 – i.e. a constant– at c. What is the best you're going to be able to do? Well, about all you can do is make sure that the function $f(x)$ and your approximation T_0 will at least agree at c – so $T_0 = f(c)$ is probably the best approximation we could hope for.



Now, how about approximating $f(x)$ by a linear polynomial? Well, we can once again at least force $T_1(c) = f(c)$ – but we have this linear term we can use as well! So, we can actually make something that agrees with $f(x)$ at c, and also has the same first derivative as $f(x)$ at c, by defining $T_1(x) = f(c) + f'(c) \cdot (x - c)$! This seems to be a pretty good approximation of what f is doing at c – not only does it agree with f there, but at c, it has the same rate of change as f!



Consider now how we could approximate f by a quadratic polynomial. As you may have guessed from the above, we're trying to find a $T_2(x)$ that will match f at its 0th, 1st, and 2nd derivatives – so how can we do that?

Well, if $T_2(c) = f(c)$, then we can write it in the form

$$T_2(x) = f(c) + \alpha_1(x - c) + \alpha_2(x - c)^2,$$

for some constants α_1 and α_2, because we know when we plug in c we should get zero!

So: if the first derivative of $T_2(x)$ is equal to $f'(x)$ at c, then we also have that

$$f'(c) = (T_2(x))' = \alpha_1 + 2\alpha_2(x - c),$$

i.e.$\alpha_1 = f'(c)$.

As well, if the second derivative agrees, then we must also have

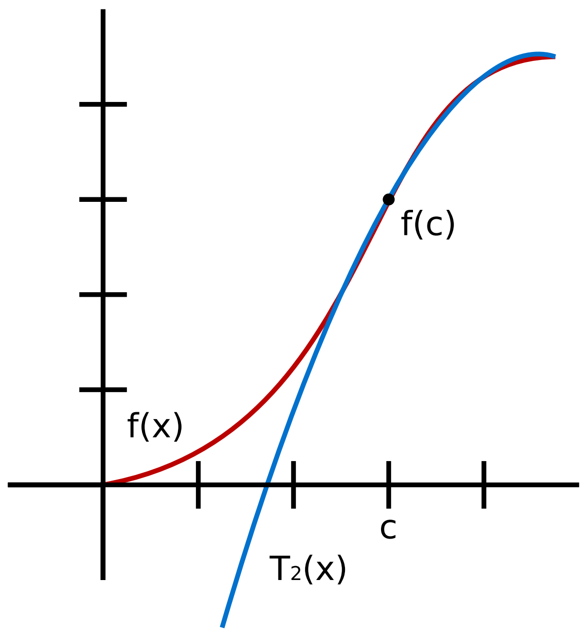
$$f''(c) = (T_2(x))'' = 2\alpha_2,$$

i.e. $\alpha_2 = f''(c)/2$.

So, we have that

$$T_2(x) = f(c) + \frac{f'(c)}{1}(x - c) + \frac{f''(c)}{2}(x - c)^2.$$

This gives us an even better local approximation of f, as T_2 has the same value, slope, and local curvature as f does at c!



In general, if we want to make a polynomial T_n of degree n that has the same first n derivatives as some function f, we just need to find constants $\alpha_0, \alpha_1, \ldots \alpha_n$ so that the polynomial

$$T_n(x) = \alpha_0 + \alpha_1(x - c) + \alpha_2(x - c)^2 + \ldots + \alpha_n(x - c)^n$$

has the same derivatives as $f(x)$ does at c.

But what do derivatives of T_n look like? Well, the 0th derivative at c is just

$$T_n(c) = \alpha_0 + \alpha_1(c - c) + \alpha_2(c - c)^2 + \ldots + \alpha_n(c - c)^n = \alpha_0,$$

the first derivative at c is just

$$\begin{aligned}
T_n'(c) &= (\alpha_0 + \alpha_1(x-c) + \alpha_2(x-c)^2 + \ldots + \alpha_n(x-c)^n)'\Big|_c \\
&= (\alpha_1 + 2\alpha_2(x-c) + \ldots + n\alpha_n(x-c)^{n-1})\Big|_c \\
&= \alpha_1 + 2\alpha_2(c-c) + \ldots + n\alpha_n(c-c)^{n-1}, \\
&= \alpha_1
\end{aligned}$$

the second derivative is just

$$\begin{aligned}
T_n''(c) &= (\alpha_0 + \alpha_1(x-c) + \alpha_2(x-c)^2 + \ldots + \alpha_n(x-c)^n)''\Big|_c \\
&= (2\alpha_2 + 3\cdot 2\alpha_3(x-c) + \ldots + n(n-1)\alpha_n(x-c)^{n-2})\Big|_c \\
&= 2\alpha_2 + 3\cdot 2\alpha_3(c-c) + \ldots + n(n-1)\alpha_n(c-c)^{n-1} \\
&= 2\alpha_2,
\end{aligned}$$

the third derivative is just (bear with me here)

$$\begin{aligned}
T_n'''(c) &= (\alpha_0 + \alpha_1(x-c) + \alpha_2(x-c)^2 + \ldots + \alpha_n(x-c)^n)'''\Big|_c \\
&= (3\cdot 2\cdot 1\alpha_3 + 4\cdot 3\cdot 2\alpha_4(x-c) + \ldots + n(n-1)(n-2)\alpha_n(x-c)^{n-3})\Big|_c \\
&= 3\cdot 2\cdot 1\alpha_3 + 4\cdot 3\cdot 2\alpha_4(c-c) + \ldots + n(n-1)(n-2)\alpha_n(c-c)^{n-3}) \\
&= 3!\alpha_3;
\end{aligned}$$

and following the same pattern, we have (inductively) that

$$T_n^{(k)}(c) = k! \cdot \alpha_k.$$

So, if these derivatives of $T_n(x)$ are suppose to agree with $f(x)$'s derivatives at c, we then have that $f^{(k)}(c) = k! \cdot \alpha_k$, for every k – i.e that $\alpha_k = f^{(k)}(c)/k!$, and thus that

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} \cdot (x-c)^k.$$

This polynomial that we just constructed has its first n derivatives in agreement with f at c! Consequently, it's a pretty good approximation for f around c – good enough, in fact, it has a name! This is called the n-th degre Taylor polynomial for f at c, and we'll denote such a polynomial by $T_n(f)(x)$.

3. Taylor Polynomials - Examples

So: if you don't care about the why, the short of the above is that for a function f with n derivatives at c, we can make a n-th degree polynomial to approximate f near c. We call it the Taylor polynomial, and define it as

$$T_n(f)(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} \cdot (x-c)^k.$$

We calculate a few examples below:

Question 3.1. *If $f(x) = a^x$, prove that the Taylor series for f around 0 is*

$$T_n(f)(x) = \sum_{k=0}^{n} \frac{(\log(a))^k}{k!} x^k.$$

Proof. So: we proceed by induction, which is pretty much the only way to prove that a given function has a set Taylor series. Specifically, we claim that the k-th derivative of a^x is $(\log(a))^k \cdot a^x$ – if we can show that this is true, then we would have that

$$\begin{aligned} T_n(f)(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \\ &= \sum_{k=0}^{n} \frac{(\log(a))^k \cdot a^0}{k!} x^k \\ &= \sum_{k=0}^{n} \frac{(\log(a))^k}{k!} x^k \end{aligned}$$

by the definition of the Taylor polynomial, and we'd be done!

So: again, we are trying to prove that the k-th derivative of a^x is $(\log(a))^k \cdot a^x$.

Base case: $k = 0$. This is trivial, as $a^x = (\log(a))^0 \cdot a^x$.

Inductive step: Suppose that the k-th derivative of a^x was $(\log(a))^k \cdot a^x$. Then

$$\begin{aligned} \frac{d^{k+1}}{dx^{k+1}}(a^x) &= \frac{d}{dx}\left(\frac{d^k}{dx^k}(a^x)\right) \\ &= \frac{d}{dx}\left((\log(a))^k \cdot a^x\right) \\ &= (\log(a))^k \cdot (a^x)' \\ &= (\log(a))^k \cdot (e^{x\log(a)})' \\ &= (\log(a))^{k+1} \cdot e^{x\log(a)} \\ &= (\log(a))^{k+1} \cdot a^x. \end{aligned}$$

This proves our inductive claim, and thus shows (again, by the definition of the Taylor polynomial, that

$$T_n(a^x) = \sum_{k=0}^{n} \frac{(\log(a))^k}{k!} x^k.$$

Note that when $a = e$, this gives us the very special Taylor series for e of

$$T_n(e^x) = \sum_{k=0}^{n} \frac{1}{k!} x^k,$$

as $\log(e) = 1$. $\square$

Question 3.2. *If $f(x) = (x+1)^a$, show that the Taylor series of f around 0 is given by*

$$T_n(f)(x) = \sum_{k=0}^{n} \binom{a}{k} x^k,$$

where $\binom{a}{k}$ is the binomial coefficient defined by

$$\binom{a}{k} = \frac{a \cdot (a-1) \cdot (a-2) \ldots \cdot (a-k+1)}{k!}, \; k \leq a$$

Proof. So: just like before, we want to prove by induction that

$$\frac{d^k}{dx^k}\left((x+1)^a\right) = (a \cdot (a-1) \cdot (a-2) \ldots \cdot (a-k+1)) \cdot (x+1)^{a-k},$$

as this means that (by the definition of the Taylor polynomial)

$$\begin{aligned} T_n(f)(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \\ &= \sum_{k=0}^{n} \frac{(a \cdot (a-1) \cdot (a-2) \ldots \cdot (a-k+1)) \cdot (0+1)^{a-k}}{k!} x^k \\ &= \sum_{k=0}^{n} \frac{(a \cdot (a-1) \cdot (a-2) \ldots \cdot (a-k+1))}{k!} x^k \\ &= \sum_{k=0}^{n} \binom{a}{k} x^k. \end{aligned}$$

So: we proceed by induction.

Base case:$k = 0$. This is trivial, as the product $a \cdot \ldots (a-k+1)$ is empty when $k = 0$, making our claim just that $(x+1)^a = (x+1)^a$.

Inductive step: Assume that

$$\frac{d^k}{dx^k}\left((x+1)^a\right) = (a \cdot (a-1) \cdot (a-2) \ldots \cdot (a-k+1)) \cdot (x+1)^{a-k}.$$

Then, we have that

$$\begin{aligned}
\frac{d^{k+1}}{dx^{k+1}}\left((x+1)^a\right) &= \frac{d}{dx}\left(\frac{d^k}{dx^k}\left((x+1)^a\right)\right) \\
&= \frac{d}{dx}\left((a\cdot(a-1)\cdot(a-2)\ldots\cdot(a-k+1))\cdot(x+1)^{a-k}\right) \\
&= (a\cdot(a-1)\cdot(a-2)\ldots\cdot(a-k+1))\cdot(a-k)\cdot(x+1)^{a-k-1} \\
&= (a\cdot(a-1)\cdot(a-2)\ldots\cdot(a-k+1))\cdot(a-(k+1)-1)\cdot(x+1)^{a-(k+1)}
\end{aligned}$$

and thus we've proven our claim by induction. As noted above, this then tells us that

$$T_n\left((x+1)^a\right) = \sum_{k=0}^{n}\binom{a}{k}x^k,$$

as claimed. □

This is pretty much how all Taylor polynomial proofs work! I.e. all we did here is inductively prove that the derivatives of these functions are of a certain form, evaluated these derivatives at the point we are taking the Taylor series around, and plugged these values into the formula for the Taylor polynomial. It's just induction and derivatives – nothing we haven't done a hundred times before!

4. Taylor Polynomials and the Error Function

In our motivational discussion earlier, we described the Taylor polynomial as an attempt to "approximate" a function locally. This notion – of Taylor polynomials as approximations to a function f – motivates the following question: How "good" of an approximation is a Taylor polynomial $T_n(f)$ for a given function f?

To help answer this question, we define the n-th error function $E_n(f)(x)$ at some point c for a function $f(x)$ to be

$$E_n(f)(x) = f(x) - T_n(f)(x),$$

namely the difference between the function and its n-th order Taylor polynomial approximation at c. (This is also called the n-th remainder function by many people.)

If the function $E_n(f)(x)$ is small, we know that the n-th order Taylor polynomial is doing a "good" job of approximating our function, as the distance from it and the function itself is small – conversely, if $E_n(f)(x)$ blows up, our Taylor polynomial must be very far from $f(x)$, and thus is a poor approximation. Thus, if we want to approximate a function by a Taylor polynomial over a certain interval, all we have to do is two things:

- calculate its Taylor polynomial, and
- find the maximum of the error function over that interval.

This then will tell us that our Taylor polynomial is at least within this maximum distance from our function throughout the interval! In particular, if the error function is very small through the entire interval, we can often replace our function with the Taylor polynomial and still be accurate to a very high degree of precision.

So: we present without proof a useful identity from Apostol/class, that we will often use to bound the error function:

Proposition 4.1. *Suppose that f has a continuous $n+1$-th derivative in the interval $[a, b]$, and c is some point in $[a, b]$. Suppose further that this $n + 1$-th derivative is bounded on $[a, b]$ – i.e. that there are values m, M such that $m \leq f^{(n_1)} \leq M$.*

Then so is f's error function! Explicitly, the following inequalities holds over $[a, b]$:

$$m \cdot \frac{(x-a)^{n+1}}{(n+1)!} \leq E_n(f)(x) \leq M \cdot \frac{(x-a)^{n+1}}{(n+1)!}, \qquad x > a,$$
$$m \cdot \frac{(a-x)^{n+1}}{(n+1)!} \leq (-1)^{n+1} E_n(f)(x) \leq M \cdot \frac{(a-x)^{n+1}}{(n+1)!}, \qquad x < a.$$

An example of how this is used might make the above discussion more illuminating:

Question 4.2. *Approximate the integral*

$$\int_0^{1/3} e^{-t^2} dt$$

to within 10^{-6}.

Proof. (Denote $E(x) = e^x$ for notational convenience.)

So: from our earlier work, we know that the n-th order Taylor polynomial for e^x around 0 is given by

$$T_3(E)(x) = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3!}$$

and thus that the 3rd error function for e^x around 0 is

$$\begin{aligned} E_3(E)(x) &= e^x - T_3(E)(x) \\ &= e^x - 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3!} \end{aligned}$$

Rearranging the above tells us that

$$\begin{aligned} e^x &= 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3!} + E_3(E)(x) \\ \Rightarrow e^{-t^2} &= 1 - \frac{t^2}{1} + \frac{t^4}{2} - \frac{t^6}{3!} + E_3(E)(-t^2) \\ \Rightarrow \int_0^{1/3} e^{-t^2} dt &= \int_0^{1/3} \left(1 - \frac{t^2}{1} + \frac{t^4}{2} - \frac{t^6}{3!} + E_3(E)(-t^2)\right) dt \\ &= \left(t - \frac{t^3}{3} + \frac{t^5}{10} - \frac{t^7}{42}\right)\Bigg|_0^{1/3} + \int E_3(E)(-t^2) dt \\ &= \frac{1}{3} - \frac{1}{3^4} + \frac{1}{3^5 \cdot 2 \cdot 5} - \frac{1}{3^8 \cdot 2 \cdot 7} + \int E_3(E)(-t^2) dt \\ &= \frac{147604}{459270} + \int E_3(E)(-t^2) dt. \end{aligned}$$

So, if we can bound the integral of the error function to be smaller than 10^{-6}, then we know that the integral of e^{-t^2} is within 10^{-5} of $\frac{147604}{459270}$, and we would be done!

We set about doing this by using our proposition from earlier. Because e^x is infinitely differentiable, we know that its 4th derivative exists – explicitly, it's just e^x again! On the interval $[-1/9, 0]$, we know that this is bounded below by 0 and above by $e^0 = 1$; consequently, we have that

$$\begin{aligned} & 0 \cdot \frac{x^4}{4!} \leq E_n(E)(x) \leq 1 \cdot \frac{x^4}{(4)!} \\ \Leftrightarrow \qquad & 0 \leq E_n(E)(x) \leq \frac{x^4}{24}. \end{aligned}$$

Plugging in $-t^2$ for x then gives us that

$$0 \leq E_n(E)(-t^2) \leq \frac{t^8}{24}, t \in [0, 1/3];$$

thus, we can bound the integral $\int_0^{1/3} E_3(E)(-t^2)dt$ below by 0 and above by

$$\int_0^{1/3} \frac{t^8}{24} dt = \left. \frac{t^9}{216} \right|_0^{1/3} = \frac{1}{4251528} < .000001 = 10^{-6}.$$

So we're done! $\square$