RECITATION 4: DERIVATIVES

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1. Last Week's Homework

So: last week's HW went rather well on the whole, with an average slightly south of 90. There was only one concept that caused widespread issues, and that was the concept of discontinuity; i.e. in question (2), where you were asked to find a function with precisely one point of continuity, almost all of you found such a function, and showed that it was continuous at this one point. However, showing that it was discontinuous at all other points didn't always go so well; so I've listed a few ways to go about showing that functions are discontinuous.

Proposition 1.1. *A function f is discontinuous at a if there are sequences $\langle x_i \rangle$, $\langle y_i \rangle$ that both converge to a, such that*

$$\lim_{i \to \infty} f(x_i) \neq \lim_{i \to \infty} f(y_i).$$

Remark 1.2. By the definition above, we can easily see that the function

$$f(x) = \begin{cases} x, x \in \mathbb{Q} \\ 0 \text{ o/w} \end{cases}$$

is discontinuous at any $a \neq 0$, as we can simply pick sequences $\langle x_i \rangle$ of rational numbers and $\langle y_i \rangle$ of irrational numbers that both converge to a, and observe that $\lim f(x_i) = a \neq 0 = \lim f(y_i)$.

Proposition 1.3. *A function f is discontinuous at a if*

$$\exists \epsilon > 0 \forall \delta > 0 \exists x \in (a - \delta, a + \delta) \text{ s.t. } |f(x) - f(a)| > \epsilon.$$

Remark 1.4. This concept of discontinuity is just the straightforward negation of the standard definition of continuity, i.e. the negation of the sentence

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in (a - \delta, a + \delta) |f(x) - f(a)| < \epsilon.$$

(This is a specific example of how negation and quantifiers work: i.e., if we have any logical statement S, then we can write the negation of S, $\neg S$, by simply switching all of the quantifiers (i.e. replacing all of the $\exists$ with $\forall$ and all of the $\forall$ with $\exists$) and negating the conclusion.)

Anyways: given this definition, we can again show that the function

$$f(x) = \begin{cases} x, x \in \mathbb{Q} \\ 0 \text{ o/w} \end{cases}$$

is discontinuous at any $a \neq 0$.

To do this, fix such an a.

Let $\epsilon = |a/2|$, and choose any $\delta > 0$.

Then, if $a \in \mathbb{Q}$, we can pick (by the density of irrational numbers) an irrational number $x \in (a - \delta, a + \delta)$, and thus conclude that $|f(x) - f(a)| = |a| > |a/2| = \epsilon$.

If $a \notin \mathbb{Q}$, we can pick (by the density of rational numbers) a rational number $x \in (a, a+\delta)$; then $|f(x) - f(a)| > |a| > |a/2| > \epsilon$.

2. Derivatives: Definitions and Properties

Definition 2.1. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. If this limit does exist, we refer to it as $f'(a)$.

Remark 2.2. The following useful properties about derivatives are known:

- For all $x \in \mathbb{R}$, $(x^n)' = nx^{n-1}$, $(sin(x))' = cos(x)$, $(cos(x))' = -sin(x)$, $(c)' = 0$.
- The product rule: $(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$, if $f'(x), g'(x)$ both exist.
- The quotient rule: $f(x)/g(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$, if $f'(x), g'(x)$ both exist. (This is just the product rule applied to $f(x), g^{-1}(x)$, and so I usually don't bother remembering it.)
- The chain rule: $(f(g(x)))' = f'(g(x))g'(x)$, if $f'(x), g'(x)$ both exist.
- Rolle's theorem: if there exist $a < b$ such that $f(a) = f(b)$ and f is continuous on $[a, b]$, differentiable on (a, b), then there is an $x \in (a, b)$ such that $f'(x) = 0$.
- Mean-Value Theorem: if f is continuous on $[a, b]$ and differentiable on (a, b), then there exists an $x \in (a, b)$ such that
$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

3. Derivatives: Examples

Proposition 3.1. *The function* $\arcsin(x)$ *is differentiable on* $(-1, 1)$*, and has derivative* $\frac{1}{\sqrt{1-x^2}}$

Proof. Let $y = arcsin(x)$; then $sin(y) = x$. Taking the derivative of both sides gives us

$$\frac{d}{dx} sin(y) = \frac{d}{dx} x \Leftrightarrow$$
$$\cos(y) \cdot \frac{dy}{dx} = 1 \Leftrightarrow$$
$$\frac{dy}{dx} = fracd(\arcsin(x))dx = \frac{a}{\cos(\arcsin(x))},$$

and $\cos(\arcsin(x)) = \sqrt{1 - x^2}$ via the Pythagorean theorem and some basic trigonometry. □

Exercise 3.2. Is there a function f that is differentiable at exactly one point?

Exercise 3.3. Define the functions $f_n(x)$ iteratively as follows:

- Let $f_0(x) = x$.
- Let
$$f_n(x) = \begin{cases} .5 \cdot f_n(3x) & x \in [0, 1/3] \\ .5 & x \in [1/3, 2/3] \\ .5 + .5 \cdot f_n(3x - 2) & x \in [2/3, 1] \end{cases}$$

Let $f(x) := \lim_{n\to\infty} f_n(x)$.

Then show that this function is a continuous function on $[0,1]$ such that $f'(x) = 0$ for all x where it has a derivative, $f(0) = 0$, and $f(1) = 1$.

Alternately, show that this function is equal to the function $f = f_1 \circ f_2$, for

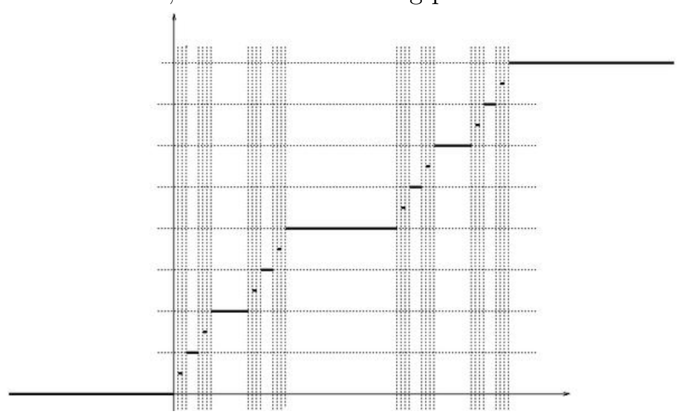
$$f_1(.x_0x_1x_2\ldots) = .y_0y_1y_2\ldots,$$

that sends a number $x \in [0,1]$ written out in its ternary (i.e. base 3) expansion to the **binary** number $.y_0y_1y_2\ldots$ in $[0,1]$ such that $y_i = 1$ if and only if $x_i = 2$ (i.e. $f_1(.012201120) = .001100010$), and

$$f_2(.x_0x_1x_2\ldots) = .x_0'x_1'\ldots x_n'200000000\ldots,$$

where $n+1$ is the smallest number such that $x_{n+1} = 1$, and proceed from there.

If you are still confused, look at the following picture:



A picture of our function.