RECITATION 3: CONTINUITY

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1. Remarks on Last Week's Homework

The homeworks from last week had a lower average than the one that preceded it, but on the whole the work submitted was fairly solid. Some comments:

- In the proof that
$$\lim_{n\to\infty} n^3/(n+1)^3 = 1,$$
many of you successfully found valid choices of N, usually $N > 3/\epsilon$ (though there were several other acceptable choices). However, finding this N is half of the proof: once it is acquired, you have to then demonstrate that for any $n > N$, $|n^3/(n+1)^3 - 1|$ is smaller than ϵ. Not doing this lost several people a healthy chunk of points.
- For the second question, you were tasked with showing that
$$2 < (1 + 1/n)^n,$$
which many of you did using your earlier work; an acceptable and far faster way would be to use Bernoulli's formula,
$$1 + nx < (1 + x)^n,$$
which you proved last week.
- When looking at
$$\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}},$$
many of you attempted to use the Alternating Series Test to show this does not converge. This is inapplicable here, as the Alternating Series Test simply says that any series that satisfies its two conditons will converge – it says **nothing** about sequences that do not satisfy its conditions. The test to have used here is the limit test, or simply the definition of what a convergent series is.
- Also, when proving things, **do not** begin by assuming what you want to show. Several of you lost points for doing this.

2. Continuity: Definitions and Properties

Definition 2.1. For a function $f : \mathbb{R} \to \mathbb{R}$, we write
$$\lim_{x\to a} f(x) = L,$$
and say that the limit as f approaches a is L, if for every neighborhood B of L there is a neighborhood A of a such that $f(x) \in B$ whenever $x \in A$ and $x \neq a$. In $\epsilon - \delta$ notation, this definition can be reformulated as follows:
$$\lim_{x\to a} f(x) = L$$

if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $x \in \mathbb{R}$, $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.

Definition 2.2. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** at $a \in \mathbb{R}$ if

$$\lim_{x \to a} f(x)$$

exists and is equal to $f(a)$.

Intuitively, continuous functions are the functions you can "draw" without lifting your pencil. (We assume here a perfect artist and an infinitely fine point on their pencil, as well as a canvas the size of $\mathbb{R}^2$.)(But intuitively: functions you can draw.)

Remark 2.3. Continuous functions satisfy the following properties:

- The product and sum of any two continuous functions is again a continuous function, as is multiplication of a continuous function by a scalar. As well, if a continuous function f never takes on the value of 0 (i.e. $f(x) \neq 0, \forall x$), $1/f$ is also a continuous function.
- The indefinite integral
$$A(x) := \int_a^x f(t)dt$$
of a function f is continuous. (Note that this definition does not require f to be continuous!)
- The composition of two continuous functions is continuous.
- (Bolzano) If f is continuous, $a < b$, $f(a) < 0$, $f(b) > 0$, then there is a real number $c \in [a, b]$ such that $f(c) = 0$.

3. Examples

Proposition 3.1. *The function*

$$x \mapsto x^2$$

is continuous at 0.

Proof. So, by the definition, we know that x^2 is continuous at 0 if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$(\forall x \in \mathbb{R})|x - 0| < \delta \Rightarrow |x^2 - 0^2| \leq \epsilon.$$

Let $\delta = \sqrt{\epsilon}$; then, we have that

$$(\forall x \in \mathbb{R})|x| < \delta = \sqrt{\epsilon} \Rightarrow |x||x| = |x^2| \leq \epsilon.$$

So x^2 is indeed continuous at 0. □

Proposition 3.2. *The function x^3 is continuous on all of $\mathbb{R}$.*

Proof. This will hold if and only if for every $a \in \mathbb{R}$, $\epsilon > 0$ there is a $\delta_a > 0$ such that

$$(\forall x \in \mathbb{R})|x - a| < \delta_a \Rightarrow |x^3 - a^3| \leq \epsilon.$$

If $a = 0$, we can prove that x^3 is continuous at 0 by simply mirroring our earlier proof for x^2. So, assume $a \neq 0$.

Notice first that

$$|x^3 - a^3| = |(x - a)(x^2 + ax + a^2)| = |x - a||x^2 + ax + a^2|,$$

and also that
$$|x^2 + ax + a^2| < 7 \cdot a^3,$$
for x such that $\mathrm{d}(x, a) < a$.

So: using our above observations, we set $\delta_a = \min(\frac{\epsilon}{7a^2}, a)$.

Then we have two cases:

(1) $\delta_a = \frac{\epsilon}{7a^2}$. This gives us that
$$|x^3 - a^3| = |x - a||x^2 + ax + a^2| < \frac{\epsilon}{7a^2} \cdot |x^2 + ax + a^2| \leq \frac{\epsilon}{7a^2} \cdot 7a^2 = \epsilon,$$
because $|x - a| < \frac{\epsilon}{7a^2} \leq a$ implies $\mathrm{d}(x, a) < a$.

(2) $\delta_a = a$. In this case, we have (1) that $a \leq \frac{\epsilon}{7a^2} \Leftrightarrow \epsilon \geq 7a^3$, and (2) that $\mathrm{d}(x, a) < a$. Consequently, the following inequality holds:
$$|x^3 - a^3| = |x - a||x^2 + ax + a^2| < a \cdot 7a^2 \leq \epsilon.$$

In either case, our desired conclusion holds. So x^3 is continuous on all of $\mathbb{R}$. □

Proposition 3.3. $\sin, \cos$ *are continuous functions on* $\mathbb{R}$.

Proof. We can write $\sin(x)$ as the indefinite integral
$$\int_0^x cos(t)dt;$$
consequently, sin is continuous. (The same can be done for cos.) □

Proposition 3.4.
$$f(x) = \begin{cases} x \cdot (\frac{\cos^2(\frac{1}{x}) + \sin^3(\frac{1}{x})}{42 \cdot \pi}), x \neq 0 \\ 0, x = 0 \end{cases}$$
is continuous on all of $\mathbb{R}$.

Proof. First, notice that as $\cos, \sin, x$ and $1/x$ are continuous on $\mathbb{R} \setminus \{0\}$, we know that this function is itself continuous on $\mathbb{R} \setminus \{0\}$, as it is the composition and sum of several continuous functions. So it suffices to investigate $x = 0$.

So: observe that $x \geq f(x) \geq -x$, for every $x \in \mathbb{R}$, as $1 \geq (\frac{\cos^2(\frac{1}{x}) + \sin^3(\frac{1}{x})}{42 \cdot \pi}) \geq -1$ for every x. Then, by the squeeze theorem / two policemen theorem, we know that
$$\lim_{x \to 0} x = \lim_{x \to 0} -x = 0 \Rightarrow \lim_{x \to 0} f(x) = 0.$$
This agrees with $f(0) = 0$; so we can conclude that f is continuous on all of $\mathbb{R}$. □

Exercise 3.5. Is the function
$$f(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \notin \mathbb{Q} \end{cases}$$
continuous at any points in in $\mathbb{R}$?

Exercise 3.6. Is the function
$$f(x) = \begin{cases} 1/q, x \in \mathbb{Q}, x = p/q, p, q \in \mathbb{Z}, (p, q) = 1 \\ 0, x \notin \mathbb{Q} \end{cases}$$
continuous at any points in $\mathbb{R}$?

Exercise 3.7. Is there a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = 0$ for all $x \in \mathbb{R}$ where f has a derivative **and** $f(0) = 0$, $f(1) = 1$?