SEQUENCES! SERIES! SUMS!

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1. Disclaimer

Almost all of my interesting exercises have been shamelessly taken from courses I've taken from Professors Lászlo Bábai and Miklos Abert, to whom I owe an immense mathematical debt.

2. Remarks on Last Week's homework

All in all, far better than I would have thought. Not that this is an excuse to slack off; there's room for improved clarity in almost every homework. But the fundamental structure of proof was present in about every person's work, and most of you seemed to grasp how induction works. Just one comment: when proving things by induction, make sure to follow the following pattern:

Proposition 2.1. *$P(N)$ is true for all $N \in \mathbb{N}$.*

Proof.

Base Case: We want to show $P(1)$. (proof of $P(1)$).

Inductive Step: Assuming that $P(N)$ is true, we now want to prove that $P(N+1)$ is true. (proof that $P(N) \Rightarrow P(N+1)$). ☐

Occasionally homeworks would omit the words, and just have strings of equations; in context, I know precisely what you're doing, but in practice it doesn't constitute a proof. The rubrics we're given to grade homeworks mandate that we take points off for failing to adhere to these structures, so remember to write these things.

(Also: when I notice it, I correct spelling/grammar/word choice. No points are taken off for this; it's just a remnant from my days as an English major. Also, this happens whenever you go to submit things to journals, so it's not just me being pedantic.)

Exercise 2.2. Show that if the two series $\sum \frac{1}{a_n}$ and $\sum \frac{1}{b_n}$ converge, $a_n, b_n > 0$, that the series $\sum \frac{1}{a_n+b_n}$ must also converge. If just the series $\sum \frac{1}{a_n}$ converges, can the series $\sum \frac{1}{a_n+b_n}$ converge? Must it convege?

3. Preliminaries

So: the basic important definitions for sequences and series follow below:

Definition 3.1. A **sequence** $\langle a_i \rangle$ in a given set A is an ordered list, possibly infinite, of elements $a_i \in A$.

Definition 3.2. A sequence $\langle a_i \rangle$ is said to have a **limit** r if for every neighborhood U_r of r there is an index N such that for all $n > N$, $a_n \in U_r$.

Definition 3.3. A sequence $\langle a_i \rangle$ is said to have a **limit** r if for every $\epsilon > 0$ there is an index N such that for all $n > N$, $|a_n - r| \leq \epsilon$.

Definition 3.4. A sequence $\langle a_i \rangle$ is called **Cauchy** if for every $\epsilon > 0$ there is a N such that for all $n_1, n_2 > N$ $|a_{n_1} - a_{n_2}| \leq \epsilon$.

Remark 3.5. Definition 4.2 is equivalent to definition 4.3 is equivalent to asking that the sequence $\langle a_i \rangle$ is Cauchy (for all sequences of real numbers; this holds in a lot of other spaces too, but $\mathbb{R}$ is really where we'll focus for now.) It is a worthwhile exercise to prove these equivalences, if you have not done so in class.

Definition 3.6. A **series** $\sum_{i=1}^{n} a_i$ is a sum of a sequence of terms in some set in which addition is defined. n, here, can be infinity, in which case we say that

$$\sum_{i=1}^{\infty} a_i = \lim_{n \to \infty} \sum_{i=1}^{n} a_i.$$

If the above limit exists and is equal to some $r \in \mathbb{R}$, we say that the series **converges** to r.

Definition 3.7. We say a series $\sum_{i=1}^{n} a_i$ **converges absolutely** if

$$\sum_{i=1}^{\infty} |a_i|$$

converges.

Exercise 3.8. Show that in any sequence $\langle a_i \rangle$ of length $n^2 + 1$, there is either a nonincreasing or a nondecreasing subsequence of length $n + 1$. (A **subsequence** $\langle a'_i \rangle$ of a sequence $\langle a_i \rangle$ is a subset of the set $\{a_i : i \in \mathbb{N}\}$ such that for any a_i‘, a'_{i+1} in our subsequence there are elements a_j, a_k in our sequence such that $a'_i = a_j, a'_{i+1} = a_k, j < k$.)

4. Useful Properties

Remark 4.1. The following useful properties hold for limits: (Let $\langle a_i \rangle$ be a sequence with limit A, $\langle b_i \rangle$ be a sequence with limit B, let $\langle c_i \rangle$ be a sequence (possibly without a limit,) and let $\sum_{i=1}^{\infty} d_i$, $\sum_{i=1}^{\infty} f_i$, be a pair of series.)

- $\lim_{n\to\infty}(a_n + b_n) = A + B$.
- $\lim_{n\to\infty}(a_n b_n) = AB$.
- $\lim_{n\to\infty}(\frac{a_n}{b_n}) = A/B$, if B, b_i are all nonzero.
- If $A < B$, then there exists a N such that for all $n > N$, $a_n < b_n$.
- As remarked above, $\langle c_i \rangle$ has a limit if and only if it is Cauchy.
- (Weierstrass) If $\langle c_i \rangle$ is nondecreasing, $\langle c_i \rangle$ has a limit if and only if it is bounded above.
- (Bolzano-Weierstrass) Any bounded infinite sequence of real numbers has a convergent subsequence.
- (Two Policemen Theorem) If $a_i \leq b_i \leq c_i$ for all i and $A = B$, then the sequence $\langle c_i \rangle$ has a limit $C = A = B$.
- If there is a N such that for all $n > N$ $d_n \leq f_n$, then the divergence of $\sum_{i=1}^{\infty} d_i$ implies the divergence of $\sum_{i=1}^{\infty} f_i$, and the convergence of the series $\sum_{i=1}^{\infty} f_i$ implies the convergence of the series $\sum_{i=1}^{\infty} d_i$.
- (Weierstrass M-test) If there is a N such that for all $n > N$ $|d_n \leq f_n$, then $\sum_{i=1}^{\infty} d_i$ will converge whenever $\sum_{i=1}^{\infty} f_i$ converges.

- Let $d = \lim_{n\to\infty} \sqrt[n]{|d_n|}$ (if this exists.) Then if $d < 1$, the series $\sum_{i=1}^{\infty} d_i$ converges absolutely, and if $d > 1$ the series diverges: if $d = 1$ this test tells us nothing.
- Let $d = \lim_{n\to\infty} |\frac{d_n}{d_n+1}|$. If d exists and is less than 1, then $\sum_{i=1}^{\infty} d_i$ converges absolutely; if d ¿ 1, $\sum_{i=1}^{\infty} d_i$ diverges; if none of these cases hold, then this test tells us nothing.
- The series $\sum_{i=1}^{\infty} \frac{1}{n^p}$ diverges for $p \leq 1$ and converges for $p > 1$.

Remark 4.2. If you have proven any of the above results in class, they are fair game on your HW. If you have not, please prove them before using them. If you are unsure as to how to prove them, ask me for ideas.

Exercise 4.3. Can you cover the plane $\mathbb{R}^2$ with disks of positive radius D_i such that any two disks D_i, D_j contain at most one point in their intersection?

5. Examples

Examples 5.1. The following series converge:

$$\sum_{n=1}^{\infty} \frac{3}{n^{1.5}}$$
$$\sum_{n=1}^{\infty} \frac{53^n}{n!}$$
$$\sum_{n=1}^{\infty} \frac{sin(n)}{n^2}$$
$$\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$
$$\sum_{n=1}^{\infty} \frac{ln(n)}{n^2}$$

Examples 5.2. The following series diverge:

$$\sum_{i=1}^{\infty} \frac{1}{ln(n)}$$
$$\sum_{i=1}^{\infty} \frac{1}{n}$$
$$\sum_{i=1}^{\infty} \frac{n}{n+53}$$
$$\sum_{i=1}^{\infty} \frac{1}{\sqrt{n}}$$

Exercise 5.3. Suppose that the two series $\sum \frac{1}{a_n}$ and $\sum \frac{1}{b_n}$ diverge. Must the series $\sum \frac{1}{a_n+b_n}$ diverge? If so, prove this; if not, construct a counterexample.