Midterm Study Guide Math 34A

All practice problems were written to be doable WITHOUT a calculator.

Chapter 1

Practice Problem:
Use substitution to show that the following equality is FALSE. In other words, show that by substituting an appropriate number for x, the following two fractions are not equal.

$$\frac{5+x^2}{3x} = \frac{5+x}{3}$$

This is a common mistake in 34A when trying to simplify an answer.

Solution:

If we let $x = 2$, then $\frac{5+x^2}{3x} = \frac{9}{6} = \frac{3}{2}$ and $\frac{5+x}{3} = \frac{7}{3}$, so clearly the equality is false. Note that if we had chosen $x = 1$, we would have gotten both sides to equal $\frac{1}{2}$. However, as the equality fails for at least one value of x, the equality is false.

Practice Problem:
Simplify: $\frac{x^2-4}{x^2+3x+2}$

Solution:

$$\frac{x^2-4}{x^2+3x+2} = \frac{(x+2)(x-2)}{(x+2)(x+1)} = \frac{x-2}{x+1}$$

Practice Problem:
Simplify: $\left(\left(\frac{1}{4}-\frac{1}{5}\right)\times\left(\frac{1}{4}\right)^{-1}\right)^{-2}$

Solution:

$$\left(\left(\frac{1}{4}-\frac{1}{5}\right)\times\left(\frac{1}{4}\right)^{-1}\right)^{-2} = \left(\left(\frac{1}{4}-\frac{1}{5}\right)\times 4\right)^{-2} = \left(\frac{1}{20}\times 4\right)^{-2} = \left(\frac{1}{5}\right)^{-2} = 5^2 = 25$$

Practice Problem:
Solve for x in terms of y:

$$\frac{x+3}{2} + y = 2$$

Solution:

Solving for x:

$$\begin{aligned} \frac{x+3}{2} + y &= 2 \\ \Rightarrow (x+3) + 2y &= 4 \\ \Rightarrow x + 2y &= 1 \\ \Rightarrow x &= 1 - 2y \end{aligned}$$

Practice Problem:
Solve for x:
$\frac{ax+2}{2x+3} = 2$

Solution:
Solving for x:

$$\begin{aligned} \frac{ax+2}{2x+3} &= 2 \\ \Rightarrow ax+2 &= 2(2x+3) \\ \Rightarrow ax+2 &= 4x+6 \\ \Rightarrow ax-4x &= 4 \\ \Rightarrow x(a-4) &= 4 \\ \Rightarrow x &= \frac{4}{a-4} \end{aligned}$$

Practice Problem:
Solve the following system of equations for x and y:
$x + 5y = a$
$2x + 3y = 3$

Solution:
Multiplying the first equation by -3 gives: $-3x - 15y = -3a$.
Multiplying the second equation by 5 gives: $10x + 15y = 15$.
Adding them together then yields: $7x = 15 - 3a$, so $x = \frac{15-3a}{7}$.
Plugging this into the first equation then gives:

$$\begin{aligned} \frac{15-3a}{7} + 5y &= a \\ \Rightarrow 5y &= a - \frac{15-3a}{7} \\ \Rightarrow 5y &= \frac{7a}{7} - \frac{15-3a}{7} \\ \Rightarrow 5y &= \frac{7a-15+3a}{7} \\ \Rightarrow 5y &= \frac{10a-15}{7} \\ \Rightarrow y &= \frac{2a-3}{7} \end{aligned}$$

Practice Problem:
What is x% of 7 as a percentage of 31?

Solution:
$x\%$ of 7 is $\dfrac{x}{100} \times 7 = \dfrac{7x}{100}$.

This number as a percentage of 31 is then $\dfrac{\frac{7x}{100}}{31} \times 100\% = \dfrac{7x}{31}\%$

Practice Problem:
Find the inverse to the function $f(x) = 1 + \dfrac{7}{x^3 - 2}$.

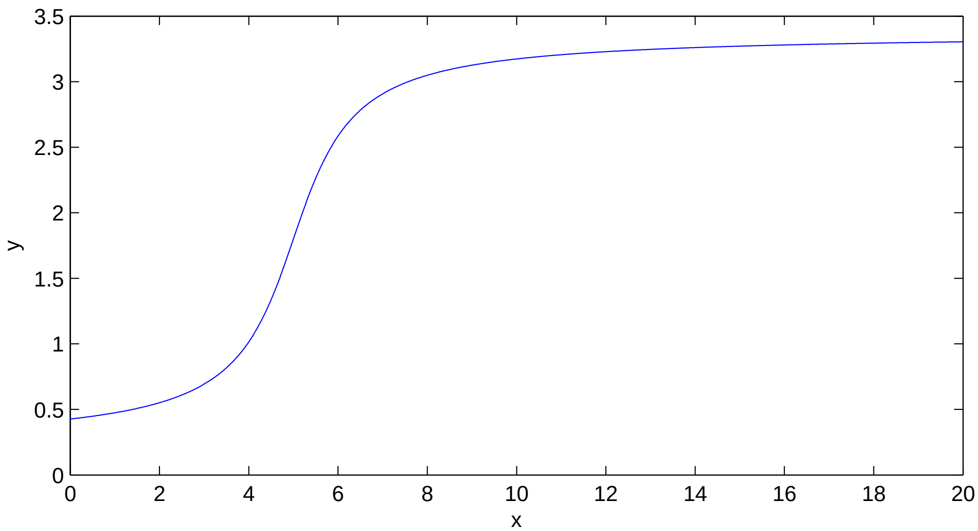
Solution:
Let $y = f(x)$. So $y = 1 + \dfrac{7}{x^3 - 2}$ and solve for x:

$$\begin{aligned} y &= 1 + \frac{7}{x^3 - 2} \\ \Rightarrow y - 1 &= \frac{7}{x^3 - 2} \\ \Rightarrow x^3 - 2 &= \frac{7}{y - 1} \\ \Rightarrow x^3 &= \frac{7}{y - 1} + 2 \\ \Rightarrow x &= \sqrt[3]{\frac{7}{y - 1} + 2} \end{aligned}$$

Therefore, $f^{-1}(x) = \sqrt[3]{\dfrac{7}{x - 1} + 2}$.

Chapter 2

Practice Problem:
Suppose this is the graph of $f(x)$.



- What is $f^{-1}(1)$?
- Is $f(x)$ increasing more rapidly when $x = 5$ or $x = 10$?

Solution:

- $f^{-1}(1) = 4$
- $f(x)$ is increasing more rapidly when $x = 5$ since the tangent line there has steeper positive slope than at $x = 10$.

Chapter 4

1. Unit conversions (know time and metric conversions)
2. How area and volume grow with respect to linear dimensions

Practice Problem:
A salt water solution is being poured into a tank at a rate of 5 L / minute. The concentration of salt in this solution is 2 mg / cm^3. How many hours will it take until there are 60 grams of salt in the tank? Note that 1 L = 1000 cm^3.

Solution:

The rate that the salt is entering the tank is:

$$\frac{5\text{ L}}{1\text{ minute}} \times \frac{1000\text{ cm}^3}{1\text{ L}} \times \frac{2\text{ mg}}{1\text{ cm}^3} \times \frac{1\text{ g}}{1000\text{ mg}} = 10\text{ g/min}$$

Note how due to cancelation, we end up only having to multiply 5×2.

This rate = mass / time, so time = mass / rate. Therefore if we let mass = 60 g and the rate as we found above, we get:

$$\text{time} = \frac{\text{mass}}{\text{rate}} = \frac{60\text{ g}}{\frac{10\text{ g}}{1\text{ min}}} = 60\text{ g} \times \frac{1\text{ min}}{10\text{ g}} \times \frac{1\text{ hour}}{60\text{ min}} = \frac{1}{10}\text{ hours.}$$

Practice Problem:

Using just the facts that 1 L = 1000 cm^3 and 100 cm = 1 m, figure out how many liters are in 1 m^3.

Solution:

$$1\text{ m}^3 \times \frac{(100\text{ cm})^3}{(1\text{ m})^3} = \frac{1000000\text{ cm}^3}{1\text{ m}^3} \times \frac{1\text{ L}}{1000\text{ cm}^3} = 1000\text{ L.}$$

Practice Problem:

Suppose a storage facility offers storage pods of two different sizes. If the larger storage pods have four times the linear dimensions of the smaller storage pods, how much more could we store in the larger pods?

Solution:

As volume grows as the cube of linear dimensions, since the larger pod has linear dimensions 4 times that of the smaller pod, the volume of the larger pod will be $4^3 = 64$ times that of the smaller pod. So we can store 64 times as much stuff in the larger pod than the smaller pod.

Chapter 5

Practice Problem:

You guess that for $\frac{2}{3}$ of the days in June, the skies have been overcast, but it turns out that $\frac{3}{4}$ of the days the skies have been overcast. What is your percent error?

Solution:

$$\frac{|\frac{3}{4} - \frac{2}{3}|}{\frac{3}{4}} \times 100\% = \frac{\frac{1}{12}}{\frac{3}{4}} \times 100\% = \frac{1}{12} \times \frac{4}{3} \times 100\% = \frac{1}{9} \times 100\% = \frac{100}{9}\%$$

Practice Problem:

What is $\lim_{x\to\infty} \frac{5-3x^3}{7-2x^3}$? What is $\lim_{x\to 0} \frac{5-3x^3}{7-2x^3}$?

Solution:

Dividing by the highest power of x (which is x^3), we get:

$$\lim_{x\to\infty} \frac{5-3x^3}{7-2x^3} = \lim_{x\to\infty} \frac{\frac{5}{x^3}-3}{\frac{7}{x^3}-2} = \frac{0-3}{0-2} = \frac{-3}{-2} = \frac{3}{2}$$

$$\lim_{x\to 0} \frac{5-3x^3}{7-2x^3} = \lim_{x\to 0} \frac{5-3\cdot 0}{7-2\cdot 0} = \frac{5}{7}$$

Practice Problem:
What is $\lim_{h\to 0} \dfrac{(4+h)^2-16}{h}$?

Solution:

Note that if we immediately substitute $h=0$ we get $\lim_{h\to 0} \dfrac{0}{0}$, so we know there is some simplifying to be done.

$\lim_{h\to 0} \dfrac{(4+h)^2-16}{h} = \lim_{h\to 0} \dfrac{16+8h+h^2-16}{h} = \lim_{h\to 0} \dfrac{8h+h^2}{h} = \lim_{h\to 0}(8+h) = 8$

Practice Problem:
What is the change in $f(x) = 2^x$ as x increases from 0 to 4?

Solution:

$f(4) - f(0) = 2^4 - 2^0 = 16 - 1 = 15$

Practice Problem:
Evaluate $\sum_{n=1}^{3}\left(\sum_{m=2}^{4} \dfrac{n}{m}\right)$.

Solution:

$\sum_{n=1}^{3}\left(\sum_{m=2}^{4} \dfrac{n}{m}\right) = \sum_{n=1}^{3}\left(\dfrac{n}{2}+\dfrac{n}{3}+\dfrac{n}{4}\right) = \sum_{n=1}^{3}\left(\dfrac{6n}{12}+\dfrac{4n}{12}+\dfrac{3n}{12}\right) = \sum_{n=1}^{3}\left(\dfrac{13n}{12}\right)$
$= \dfrac{13}{12}+\dfrac{26}{12}+\dfrac{39}{12} = \dfrac{78}{12} = \dfrac{13}{2}$

Practice Problem:
Combine the following summations into a single sum:

$\sum_{n=1}^{45} a_n + \sum_{m=46}^{100} a_m - \sum_{k=1}^{55} a_k$

Solution:

$\sum_{n=1}^{45} a_n + \sum_{m=46}^{100} a_m - \sum_{k=1}^{55} a_k = \sum_{n=1}^{100} a_n - \sum_{k=1}^{55} a_k = \sum_{n=56}^{100} a_n$

Chapter 6

Practice Problem:
What is the line that passes through (2,3) and hits the x-axis at 7?

Solution:

The point corresponding to crossing the x-axis at 7 is (7,0). So the slope of the line is $m = \dfrac{3-0}{2-7} = -\dfrac{3}{5}$. Then plugging (7,0) and this slope into point-slope form yields:

$$y - 0 = \frac{-3}{5}(x-7)$$
$$\Rightarrow y = -\frac{3}{5}x + \frac{21}{5}$$

Practice Problem:
Suppose the time it takes Maggie to drive across the country is inversely proportional to her average driving speed and proportional to the number of times she stops to rest. Maggie drives across the country in 6 days, her average speed is 80 mph and she stops 4 times. How fast would Maggie have had to drive if she wanted to take 10 days and stop only 5 times?

Solution:
Let t = days to drive across the country.
Let s = average driving speed.
Let n = number of stops.
Then if k is our proportionality constant, we have that $t = \dfrac{k \cdot n}{s}$.

From the first scenario, we get that $t = 6$, $s = 80$, and $n = 4$, so $k = \dfrac{t \cdot s}{n} = \dfrac{6 \cdot 80}{4} = 120$.

Then in the second scenario, we want to find a speed when $t = 10$ and $n = 5$. So we solve for $s = \dfrac{k \cdot n}{t}$ and get that $s = \dfrac{120 \cdot 5}{10} = 60$. Thus Maggie would have had to drive at an average speed of 60 mph.

Practice Problem:
Let $f(x) = 3^x$. Use the points given by $x = 1$ and $x = 2$ and linear interpolation to estimate $3^{1.5}$.

Solution:
The points we get when we plug $x = 1$ and $x = 2$ into 3^x are (1,3) and (2,9) respectively. The slope of the line through these two points is $m = \dfrac{9-3}{2-1} = 6$. Plugging this slope and (1,3) into point-slope form then yields $y - 3 = 6(x - 1) \Rightarrow y = 6x - 3$.

We get our estimate for $3^{1.5}$ by substituting $x = 1.5$ into our line equation (as this line is attempting to "mimic" the 3^x function). We get $y = 6 \cdot 1.5 - 3 = 9 - 3 = 6$. So our estimate for $3^{1.5}$ is 6.