

UNIQUENESS OF BRIDGE SURFACES FOR 2-BRIDGE KNOTS

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ABSTRACT. We show that any 2-bridge knot in S^3 has a bridge sphere from which any other bridge surface can be obtained by stabilization, meridional stabilization, perturbation and proper isotopy.

1. INTRODUCTION

Establishing the uniqueness of Heegaard splittings for certain 3-manifolds has been an interesting and surprisingly difficult problem. One of the earliest known results was that of Waldhausen [Wa] who proved that S^3 has a unique Heegaard splitting up to stabilization. In [BoO], Bonahon and Otal proved that the same is true of lens spaces (manifolds with a genus one Heegaard surface). A later proof [RS] made use of the fact that any two weakly incompressible Heegaard splittings of a manifold can be isotoped to intersect in a nonempty collection of curves that are essential on both Heegaard surfaces.

There is an analogue to Heegaard splitting in the theory of links in 3-manifolds. (By link, we include the possibility that K has one component, i. e. a knot is a link.) Consider a link K in a closed orientable 3-manifold M with a Heegaard surface P (i.e. $M = A \cup_P B$ where A and B are handlebodies) and require that each arc of $K - P$ is P -parallel in $M - P$. We say that K is in *bridge position* with respect to P and that P is a *bridge surface* for the pair (M, K) . Beyond the philosophical analogy between Heegaard splittings for 3-manifolds and bridge surfaces for links in 3-manifolds, notice that there is also this precise connection: If P is a bridge surface for a link K in M , then the cover \hat{P} of P in the 2-fold branched cover \hat{M} of M is a Heegaard surface for the manifold \hat{M} .

Questions about the structure of Heegaard splittings on 3-manifolds often have analogies with questions about bridge surfaces. For example, it is natural to ask whether there are pairs (M, K) that have a unique bridge surface, up to some obvious geometric operations analogous to Heegaard stabilization. In [HS2] Hayashi and Shimokawa proved that this is true for bridge surfaces of the unknot. We will show that the same is true for bridge surfaces of 2-bridge knots. (And presumably for 2-bridge links as well, though we do not pursue that here, because of the technical obstacle that the theory in [STo] so far has not been explicitly extended to 3-manifolds with non-empty boundary.) This result can be viewed as the analogue for bridge surfaces of the result of Bonahon and Otal mentioned above. Our approach will be analogous to that of [RS], working from the central result of [STo]: in

Date: September 19, 2006.

1991 Mathematics Subject Classification. 57M25, 57M27, 57M50.

Key words and phrases. bridge position, Heegaard splitting, strongly irreducible, weakly incompressible, 2-bridge links.

Research of the first author is partially supported by an NSF grant.

the absence of incompressible Conway spheres, two c -weakly incompressible bridge surfaces can be properly isotoped to intersect in a non-empty collection of closed curves, each of which is essential (including non-meridional) in both surfaces.

2. DEFINITIONS AND NOTATION

If X is any subset of a 3-manifold M and K is a 1-manifold properly embedded in M , let $X_K = X - K$. A disk $D \subset M$ that meets K exactly once is called a *punctured disk*. If F is an embedded surface in M transverse to K , a simple closed curve on F_K is *essential* if it doesn't bound a disk or a punctured disk on F_K . An embedded disk $D \subset M_K$ is a *compressing disk* for F_K if $D \cap F_K = \partial D$ and ∂D is an essential curve in F_K . A *cut-disk* for F_K is a punctured disk D^c in M_K such that $D^c \cap F_K = \partial D^c$ and ∂D^c is an essential curve in F_K . A possibly punctured disk D^* that is either a cut disk or a compressing disk will be called a *c-disk* for F_K . The surface F_K is called essential if it has no compressing disks (it may have cut-disks), it is not a sphere that bounds a ball in M_K and it is not ∂ -parallel in $M - \eta(K)$ where $\eta(K)$ is a regular open neighborhood of K .

A properly embedded arc $\alpha \subset F_K$ is *inessential* if there is a disk on F_K whose boundary is the endpoint union of α and a subarc of ∂F . Otherwise α is *essential*. A *∂ -compressing disk* for F_K is an embedded disk $D \subset M$ with an interior disjoint from F_K such that ∂D is the endpoint union of an essential arc of F_K and an arc lying in ∂M .

Any term describing the compressibility of a surface can be extended to account not only for compressing disks but also c -disks. A surface in M that is transverse to K will be called *c -incompressible* if it has no c -disks. A surface F in M is called a *splitting surface* if M can be written as the union of two 3-manifolds along F . If F is a splitting surface for M , we will call F_K *c -weakly incompressible* if any pair of c -disks for F_K on opposite sides of the surface intersect. If F_K is not c -weakly incompressible, it is *c -strongly compressible*.

A properly embedded collection of arcs $T = \cup_{i=1}^n \alpha_i$ in a compact 3-manifold is called *boundary parallel* if there is a collection $E = \cup_{i=1}^n E_i$ of embedded disks, so that, for each $1 \leq i \leq n$, ∂E_i is the end-point union of α_i and an arc in the boundary of the 3-manifold. A standard cut-and-paste arguments shows that if there is such a collection, there is one in which all the disks are disjoint. If the manifold is a handlebody A , the arcs are called *bridges* and disks of parallelism are called *bridge disks*. Let M be a closed irreducible 3-manifold and let P be a Heegaard surface for M decomposing the manifold into handlebodies A and B . A link K is in bridge position with respect to P if each collection of arcs $A \cap K$ and $B \cap K$ is a collection of bridges. We say that P is a bridge surface for the pair (M, K) and the triple $(M; P, K)$ is a bridge presentation of $K \subset M$.

Two disjoint surfaces $F, S \subset M$ transverse to K will be called parallel if they cobound a product region and all arcs of the link in that region can be isotoped to be vertical with respect to the product structure. F is properly isotopic to S if there is an isotopy from F to S so that F remains transverse to K throughout the isotopy, i.e the isotopy of F_K to S_K is proper in M_K . Unless otherwise stated, all isotopies will be proper isotopies.

3. NEW BRIDGE SURFACES FROM OLD

Given a bridge surface P for (M, K) , it is easy to construct more complex bridge surfaces for (M, K) from P . There are three straightforward ways to do this. The first is easiest: simply add a trivial 1-handle to one of the handlebodies, say A . This creates a dual 1-handle in B . The new bridge surface, P' is said to be *stabilized* and it is characterized by the presence of compressing disks for P' , one in A and one in B , that intersect in exactly one point.

A second way to construct a more complicated bridge surface is almost as easy to see: Suppose there are a pair of bridge disks $E_A \subset A$ and $E_B \subset B$ so that the arcs $E_A \cap P$ and $E_B \cap P$ intersect precisely at one end. Then K is said to be *perturbed* with respect to P (and vice versa), and E_A, E_B are called *cancelling disks* for K . (This is one of two cases of the notion of “cancellable” bridges, as defined by Hayashi and Shimokawa in [HS2]. The other case occurs when a component of K is in 1-bridge position, and both bridges, and so a whole component, can be simultaneously isotoped into the bridge surface.) The word perturbed is used because one way a bridge presentation with this property can be obtained is by starting with any bridge presentation for K and perturbing K near a point of $K \cap P$, introducing a minimum and an adjacent maximum. The following lemma shows this is in some sense the only way in which a perturbed link can arise.

Lemma 3.1. *Suppose K is perturbed with respect to the bridge surface P . Then there is a knot K' in bridge position with respect to P , such that $|K' \cap P| = |K \cap P| - 2$ and K is properly isotopic to the knot obtained from K' by introducing a minimum and an adjacent maximum near a point of $K' \cap P$.*

Proof. Let E_A, E_B be the cancelling bridge disks, intersecting P in arcs α and β respectively, so that $\alpha \cap \beta = E_A \cap E_B$ is a single point $p \in P$, an end point of both α and β . A standard cut-and-paste argument shows that there is a disjoint collection of bridge disks for $K \cap A$ so that the collection contains E_A . In fact

Claim: There is a disjoint collection Δ_A of bridge disks for $K \cap A$ so that $E_A \in \Delta_A$ and $\Delta_A \cap \beta = \partial\beta$.

We begin with a disjoint collection and redefine it so as to eliminate all intersection points with the interior of β . The proof is by induction on the number of points in $\Delta_A \cap \text{interior}(\beta)$. If the intersection is empty, there is nothing to prove. Otherwise, suppose that q is the closest point of $\Delta_A \cap \beta$ to p in $\text{interior}(\beta)$, and let β' be the subsegment of β between q and p . Suppose $E' \neq E_A$ is the bridge disk containing q . Then a regular neighborhood of $E' \cup \beta' \cup E_A$ has boundary consisting of two disks - one parallel to E' and the other a new bridge disk for the bridge $E' \cap K$ that is disjoint from all other bridge disks and intersects β in one fewer point. This provides the inductive step, establishing the claim.

Following the claim, let $E' \neq E_A$ be the bridge disk in Δ_A that is incident to the opposite end of β from p ; following the claim E' , like E_A , is disjoint from the interior of β . Use E_B to (non-properly) isotope the arc $K \cap E_B$ to β and push it through P . This reduces the number of points in $K \cap P$ by two, but P is still a bridge surface for the knot. It's clear that $K \cap B$ still consists of bridges, since all we've done is remove one. The change in $K \cap A$ is to attach the bridge disk E' to E_A by a band, and the result is clearly still a disk. It's easy to see that the original positioning of K is properly isotopic to a perturbation of the new positioning of K with respect to P . \square

Here is a third way to produce a new bridge surface for (M, K) , called *meridional stabilization*. Begin with a bridge presentation $M = A \cup_P B$ of K and suppose there is a component K_0 of K that is not in 1-bridge position with respect to P . Let β be a bridge in $K_0 \cap B$ and let A' be the union of A together with a neighborhood of β . Let $P' = \partial A'$ and let B' be the closed complement of A' in M . The decomposition $M = A' \cup_{P'} B'$ is a Heegaard splitting, indeed a stabilization of $M = A \cup_P B$ since a meridian for A' dual to β intersects the remnants of a bridge disk for β in B' in a single point. Moreover, K is in bridge position with respect to P' . It is obvious that $K \cap B'$ is a collection of bridges, since $K \cap B$ was. And the new component of $K \cap A'$ has, as a bridge disk, the union of two bridge disks of $K \cap A$ attached together by a band running along β .

Lemma 3.2. *A bridge surface P' for K is meridionally stabilized if and only if there is a cut-disk in A' and a compressing disk in B' (or vice versa) that intersect in exactly one point.*

Proof. If P' is constructed by meridional stabilization, as described above, then, as we have seen, a meridian disk in A' dual to β is a cut disk for A' that intersects the remnants of a bridge disk for β in a single point.

Conversely, suppose there is a cut disk $E_A \subset A'$ for A' and a compressing disk $E_B \subset B'$ that intersect in a single point. Then P' is the stabilization of the Heegaard surface P obtained by cutting A' along E_A .

Claim: K is in bridge position with respect to P .

A standard cut and paste argument shows that the bridge disks for $K \cap B'$ can be taken to be disjoint from E_B . They can also be taken to be disjoint from ∂E_A , for any time a bridge disk for K in B' crosses ∂E_A , one can reroute it around ∂E_B , adding a copy of the disk E_B to the bridge disk, to get a bridge disk which intersects ∂E_A fewer times (see Figure 1). Once all bridge disks for $K \cap B'$ are disjoint from E_A , they persist when P' is surgered along E_A . So all components of $K \cap B$ have bridge disks, except possibly the new bridge β that is produced in B , the bit of K that runs from one copy of E_A (after the cut) to the other. But E_B itself provides a bridge disk for β .

A similar argument exhibits bridge disks in A : A standard cut and paste argument shows that there is a complete collection of bridge disks for $K \cap A'$ that intersects E_A in a single arc, running from the point $K \cap E_A$ to ∂E_A . When A' is cut apart by E_A to produce A , the bridge disk for the component of $K \cap A'$ that intersects E_A is divided by this arc into bridge disks for the two resulting components of $K \cap A$, establishing the claim.

With the claim established, it is easy to see that P' is a meridional stabilization of P along β . \square

Here is yet a fourth way to construct one bridge surface from another. It will be useful here to extend, in an obvious way, the definition of bridge surface to links in compact orientable 3-manifolds with boundary. Suppose M is a compact orientable 3-manifold. A connected closed surface $P \subset M$ is a bridge surface for $K \subset M$ if P is a Heegaard surface for M (that is, the complement of P consists of two compression bodies C_1, C_2 and $P = \partial_+ C_i, i = 1, 2$) and K intersects each complementary compression body in a collection of boundary parallel arcs.

With that clarifying extension, suppose K_- is a link (possibly empty) in a 3-manifold N that has a torus boundary component $\partial_0 N$. Let P be a bridge surface

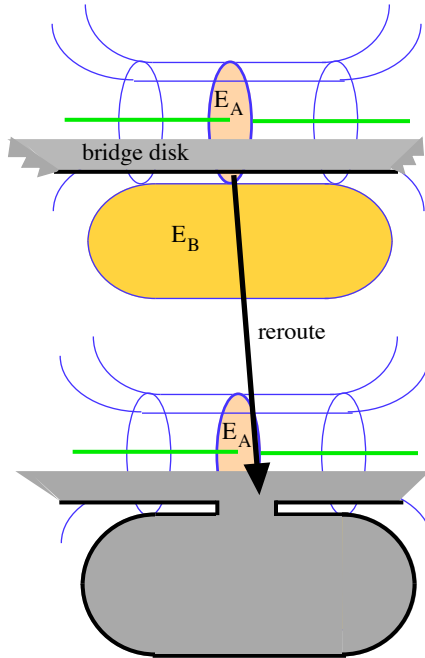


FIGURE 1

for K_- in N ; that is, P divides N into two compression bodies, and K_- intersects each of them in a collection of boundary-parallel arcs. Fill $\partial_0 N$ with a solid torus W whose core is a new curve K_0 . Then P still divides $M = N \cup_{\partial_0 N} W$ into two compression bodies and K_- still intersects each compression body in a collection of boundary-parallel arcs. Moreover, the core curve K_0 is isotopic in W to a curve on $\partial W = \partial_0 N$, so K_0 is isotopic in M rel K_- to a curve on P . Perturbing K_0 slightly makes P a bridge surface for all of $K = K_- \cup K_0$ in M . If a component of a link K in bridge position with respect to P in M can be constructed in this way, then we say that the component is *removable*.

Lemma 3.3. *Suppose P is a bridge surface for a link $K \subset M$. Then a component K_0 of K is removable if and only if K_0 can be isotoped rel $K_- = K - K_0$ so that K_0 lies on P and there is a meridian disk of one of the two compression bodies that is disjoint from K_- and intersects $K_0 \subset P$ in a single point.*

Proof. One direction is fairly straightforward: if K_0 is removable then, in the construction above, K_0 can be isotoped to a longitude of ∂W , ie to a curve in ∂W that intersects a meridian disk μ of W in a single point. That is, the wedge of circles $K_0 \vee \partial\mu \subset \partial W = \partial_0 N$. Let C be the compression body of $N - P$ on which $\partial_0 N = \partial W$ lies. Then, using the structure of the compression body, there is a proper embedding of $(K_0 \vee \partial\mu) \times I$ into $C - K_-$, with one end of $(K_0 \vee \partial\mu) \times I$ on ∂W and the other end on P . The end on P then describes an embedding of K_0 into P that intersects the meridian disk $\mu \cup (\partial\mu \times I)$ of the compression body $C \cup_{\partial_0 N} W$ in a single point.

The other direction uses the “vacuum cleaner trick”: Suppose that P is a bridge surface for a link K in M , that a component K_0 of K has been isotoped rel K_- to

lie on P , and that μ is a meridian disk for one of the complementary compression bodies C so that μ is disjoint from K_- and μ intersects K_0 in a single point. Picture the dual 1-handle to μ in C as a vacuum-cleaner hose, and use it to sweep up all of $K_0 - \eta(\partial\mu) \subset P$. Afterwards, μ is the meridian of a solid torus that is a boundary-summand of C , a solid torus for which K_0 is a longitude. Push K_0 to the core of this solid torus and remove a thin tubular neighborhood W of K_0 from the solid torus. This changes the solid torus to $\text{torus} \times I$, with the result that $C_- = C - W$ is still a compression body. Moreover, $K_- \cap C_-$ remains a collection of boundary-parallel arcs. \square

Lemma 3.4. *If a bridge surface for K is stabilized then any 1-bridge component of K is removable.*

Somewhat conversely, suppose a component K_0 for K is removable, with P , K , K_0 and meridian disk μ as defined in the proof of Lemma 3.3 above. Suppose further that there is a meridian disk λ for the other compression body so that λ is disjoint from K_- and $|\mu \cap \lambda| = 1$. Then P is stabilized.

Proof. Suppose a bridge surface P for K is obtained by stabilizing the bridge surface P' for K , and suppose K_0 is a 1-bridge component of K . Let C_1, C_2 be the compression body complementary components of P . That is, $|P \cap K_0| = |P' \cap K_0| = 2$, and P' divides K into two boundary-parallel arcs $\tau_i = C_i \cap K$, $i = 1, 2$. Let D_1, D_2 be bridge disks for τ_1, τ_2 in C_1, C_2 respectively. By general position, we can assume that the arcs $D_1 \cap P, D_2 \cap P$ have interiors that are disjoint near their end points (though there may be many intersections of their interiors away from the end points). Stabilize P' to P by attaching a 1-handle to C_2 via an arc α in D_1 near and parallel to $\tau_1 \subset \partial D_1$. Then D_2 together with the rectangle in D_1 lying between α and τ_1 describes an isotopy of K_0 to P' . A cocore of the 1-handle that was attached to C_2 is a meridian for one of the stabilized compression bodies. Via Lemma 3.3, μ exhibits K_0 as a removable component of K for the splitting surface P .

Now consider the other direction, with meridian disks $\mu \subset C_1$, $\lambda \subset C_2$, component $K_0 \subset P$ and $|K_0 \cap \mu| = 1 = |\lambda \cap \mu|$ as given in the statement of the lemma. By general position, we can assume that K_0 and λ do not intersect near μ . Move K_0 into 1-bridge position by pushing a small segment of K_0 into the interior of C_2 near μ and the interior of the rest of K_0 into the interior of C_1 . Then K_0 , hence all of K , is disjoint from both meridian disks λ and μ , which then exhibit that P is stabilized. \square

Example: Suppose K is a 2-bridge knot in S^3 and P is a Heegaard surface for the complementary 3-manifold $N = S^3 - \eta(K)$. Then either P is stabilized or it is the boundary of a regular neighborhood of the union of the knot and a single arc, and the arc is one of six standard types (see [Ko1], [Ko2], [GST]). Each of the six types of arcs (called tunnels) has the property that, once a regular neighborhood of the arc is added, then, up to isotopy, the regular neighborhood no longer depends on which 2-bridge knot we started with – indeed, we could have started with the unknot. See Figure 2. In particular, there is a meridian of the complementary handlebody that intersects a meridian disk dual to the knot in a single point. Following Lemma 3.4 we then have

Corollary 3.5. *Suppose P is any bridge surface for a 2-bridge knot $K \subset S^3$. If K is removable with respect to P , then P is stabilized.*

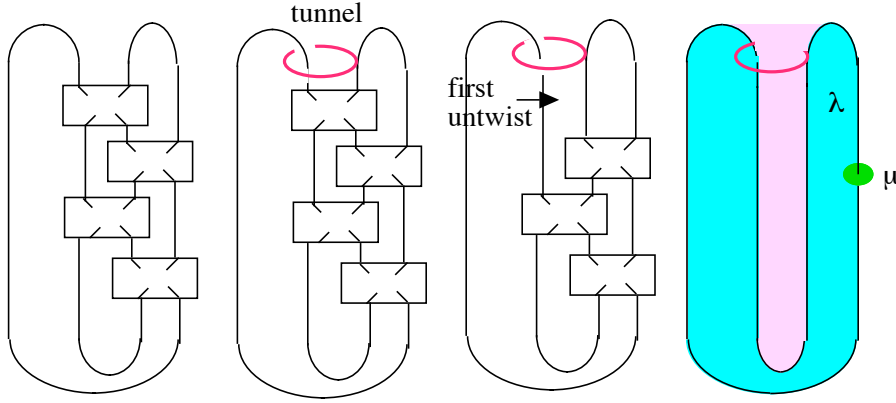


FIGURE 2

In the proof of our main theorem we will use the following already known results.

Lemma 3.6. [STo, Lemma 3.1] *Let A be a handlebody and let $(T, \partial T) \subset (A, \partial A)$ be a collection of bridges in A . Suppose F is a properly embedded surface in A transverse to T that is not a union of unpunctured disks, once-punctured disks and twice-punctured spheres. If F_T is incompressible in A_T then $\partial F \neq \emptyset$ and F_T is ∂ -compressible.*

Lemma 3.7. [STo, Lemma 3.6] *Suppose P and Q are disjoint bridge surfaces for a link $K \subset M$, decomposing M as $A \cup_P B$ and $X \cup_Q Y$ respectively. Suppose furthermore that $Q_K \subset A_K$ and P_K has a c -disk in A_K that is disjoint from Q_K , then either P_K is c -strongly compressible or $M = S^3$ and K is empty or the unknot.*

Theorem 3.8. [STo, Corollary 6.7] *Suppose P and Q are bridge surfaces for a link $K \subset M$ and P_K and Q_K are both c -weakly incompressible in M_K . If there is no incompressible Conway sphere for K in M then P_K can be properly isotoped so that P_K and Q_K intersect in a non-empty collection of curves that are essential on both surfaces.*

Theorem 3.9. [To] *Suppose, for a link $K \subset M$, M contains a c -strongly compressible bridge surface Q that is not stabilized, meridionally stabilized or perturbed. Then either*

- M contains a surface F transverse to K so that F_K is essential in M_K or
- K contains a component K_0 that is removable.

4. UNIQUE BRIDGE SURFACE

Now we will focus our attention on two-bridge links in the 3-sphere. That is, for the rest of the paper, assume $S^3 = A \cup_P B = X \cup_Q Y$, K is in bridge position with respect to both P and Q and P_K is a four times punctured sphere. In particular, henceforth A will be a ball that intersects K in two trivial arcs. The ultimate goal is to show that if K is non-trivial (i. e. neither the unknot nor the unlink of two components) and Q_K is not stabilized, meridionally stabilized or perturbed, then Q_K is also a 4-times punctured sphere properly isotopic to P_K . We will use the following technical lemma and its corollary.

Lemma 4.1. *Suppose F_K is a connected splitting surface that is properly embedded in A , so $A_K = U_K \cup_{F_K} V_K$. Further assume ∂F consists of curves that are essential in P_K , F_K is c -incompressible in V_K , but there is a ∂ -compressing disk for F_K that lies in V_K . Then F_K is parallel to a subset of P_K through V_K . In particular F_K is either an annulus or a twice punctured disk.*

Proof. Let $E \subset V_K$ be the ∂ -compressing disk for F_K . Let $\sigma = E \cap P_K$ and note that σ must be an essential arc on $P_K - F_K$ as otherwise F_K would be compressible in V_K . There are two cases to consider.

First suppose that both endpoints of σ lie on the same component of ∂F ; call this component f . As f is an essential curve on the 4-times punctured sphere P_K , it bounds two twice punctured disks on P_K , let P' be the twice punctured disk containing σ . A regular neighborhood of $P' \cup E$ consists of a copy of P' and two once punctured disks, D' and D'' , whose boundaries lie on F_K . As F_K is c -incompressible in V_K , D' and D'' each also bound once-punctured disks in F_K . Moreover, these disks must be parallel to the once-punctured disks on F_K , since twice-punctured spheres in a handlebody can only cut off trivial arcs from trivial arcs (cf [STo, Lemma 3.2]). Combining these parallelisms with the boundary compression gives a parallelism between F_K and P' .

Suppose, on the other hand, that the two endpoints of σ lie on different components of ∂F , say f and f' . As f and f' are disjoint and essential in the 4-times punctured sphere P_K , f and f' must cobound an annulus N on P_K and $\sigma \subset N$. A regular neighborhood of $N \cup E$ then consists of a copy of N and a disk D whose boundary lies on F_K . As F_K is incompressible in V_K , ∂D also bounds a disk in F_K , a disk that is parallel to D in A_K , since A_K is irreducible. Combining this parallelism with the boundary compression gives the desired parallelism between F_K and N . \square

Corollary 4.2. *Suppose F_K is a c -incompressible connected splitting surface, not an unpunctured disk, that is properly embedded in A , and suppose ∂F consists of curves that are essential in P_K . Then F_K is P_K -parallel.*

Proof. F_K can't be a once-punctured disk, since its boundary also bounds a twice-punctured disk in P_K . Since it's c -incompressible, it's incompressible, so by Lemma 3.6, F_K must be boundary-compressible. The result follows by Lemma 4.1 \square

Theorem 4.3. *Let $K \subset S^3$ be a two bridge link (not a trivial knot or link) with respect to a bridge surface $P \cong S^2 \subset S^3$. Any c -weakly incompressible bridge surface for (S^3, K) is properly isotopic to P_K .*

Proof. Suppose Q is a c -weakly incompressible bridge surface, so $S^3 = A \cup_P B = X \cup_Q Y$. P is also c -weakly incompressible. Indeed, disjoint essential curves in the 4-punctured sphere P are necessarily parallel in P_K , and so a c -strong compressing pair would provide a splitting sphere for K , contradicting the assumption that K is not a trivial link. By Theorem 3.8 we may isotope P_K so that $P_K \cap Q_K \neq \emptyset$ and all curves of $P_K \cap Q_K$ are essential on both P_K and Q_K . Furthermore assume that the number of components of intersection $|P_K \cap Q_K|$ is minimal under these restrictions. We will denote by Q_K^A and Q_K^B the surfaces $Q_K \cap A$ and $Q_K \cap B$ respectively. Similarly we will denote by P_K^X and P_K^Y the surfaces $P_K \cap X$ and $P_K \cap Y$.

Claim 1: *At least one of Q_K^A or Q_K^B has a P -parallel component.*

Q_K is not a twice-punctured sphere, since K is not the unknot. Thus there are c-disks for Q_K in both X and Y . First we will reduce to the case that there are c-disks for Q_K in both X and Y that are both disjoint from P_K .

If there aren't such c-disks, then, with no loss of generality, there is a c-disk $D_Y^* \subset Y$ for Q_K so that $|P \cap D_Y^*| > 0$ is minimal among all c-disks for Q_K in Y . If the intersection contains any simple closed curves, let α be an innermost one on D_Y^* bounding a possibly punctured disk $D_\alpha^* \subset D_Y^*$. If α were inessential in P , then a c-disk with fewer intersection curves could have been found, so α is essential in P_K . Note that as P_K is a 4-times punctured sphere and all curves of $P_K \cap Q_K$ are essential in P_K , all the curves must be parallel on P_K and are all also parallel to α . Let $N \subset P_K$ be the annulus between α and an adjacent curve of $P_K \cap Q_K$. Then by slightly isotoping the possibly punctured disk $N \cup D_\alpha^*$ we obtain c-disk for Q_K that is disjoint from P_K contradicting the choice of D_Y^* . Thus we may assume that $D_Y^* \cap P_K$ consists only of arcs. An arc of $D_Y^* \cap P_K$ that is outermost on D_Y^* cuts off a disk in Y_K that ∂ -compresses Q_K^A , say, to P_K . By Lemma 4.1, Q_K^A has a component that is P_K -parallel, establishing the claim in this case.

So now assume that there are c-disks $D_Y^* \subset Y$ and $D_X^* \subset X$ for Q_K and both are disjoint from P_K . If one disk lies in A and the other in B , then the disks would have disjoint boundaries, contradicting the assumption that Q_K is c-weakly incompressible. So these c-disks must lie on the same side of P_K . Suppose without loss that they both lie in A . Then, since Q_K is c-weakly incompressible, Q_K^B must be c-incompressible in B . But by Corollary 4.2, this implies that Q_K^B has a P_K -parallel component, again establishing the claim.

Following the claim, suppose with no loss of generality that Q_K^A has a P_K -parallel component. In this case Q_K^A must be connected, for otherwise a component of Q_K^A could be isotoped across P_K reducing $|P_K \cap Q_K|$. As all components of $P_K - Q_K$ are annuli or twice punctured disks, Q_K^A is also either an annulus or a twice punctured disk. Without loss of generality, assume Q_K^A is parallel to P_K^X (through the region $A \cap X$). See Figure 3.

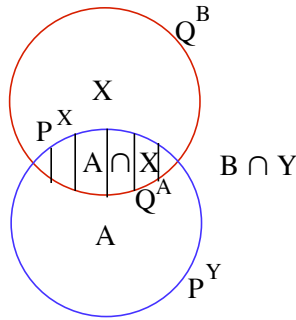


FIGURE 3

Suppose Q_K^B were c-compressible into Y_K with a c-disk D^* . Isotope P_K^X across Q_K^A so that $P_K \subset Y_K - D^*$. By Lemma 3.7 this would imply that Q_K is c-strongly compressible, a contradiction to our hypothesis. Thus we conclude that Q_K^B is either c-incompressible or c-compresses only into X_K . A similar argument with the roles of P and Q switched shows that P_K^Y does not c-compress into B_K .

Case 1: Q_K^A consists of a single P -parallel twice punctured disk.

This in particular implies that both P_K^X and P_K^Y consist of single twice-punctured disks.

Suppose first that P_K^Y is c -compressible in Y with c -disk D^* . As already shown $D^* \subset A$. Without loss of generality we may assume $\partial D^* = (P_K \cap Q_K)$ so D^* is also a c -disk for Q_K^A lying in Y_K . As Q_K is c -weakly incompressible, Q_K^B is either c -incompressible or also c -compresses in Y_K . As we have already eliminated the later option, Q_K^B must be c -incompressible and so by Corollary 4.2 Q_K^B is P_K -parallel. Thus Q_K^B is a twice punctured disk so Q_K is also a 4-times punctured sphere. In summary, if P_K^Y is c -compressible, then Q_K is also a 4-times punctured sphere and Q_K^B is c -incompressible. So, by possibly switching the names of P and Q , we may henceforth assume that P_K^Y is c -incompressible.

As P_K^Y is c -incompressible, by Lemma 3.6 it must be ∂ -compressible. Let E be the boundary compressing disk and note that $E \cap Q_K$ is an arc essential on $Q_K - P_K$ as otherwise P_K^Y would be compressible. Thus, by changing our point of view, we can conclude that Q_K^A or Q_K^B is ∂ -compressible in A or B respectively to P_K^Y .

Suppose Q_K^B is ∂ -compressible to P_K^Y . As Q_K^B is c -incompressible in Y , Lemma 4.1 implies that Q_K^B is parallel to P_K^Y . Combining this with parallelism between Q_K^A and P_K^X gives the desired isotopy between P_K and Q_K .

Suppose Q_K^A is ∂ -compressible into P_K^Y . Since P_K^Y is a c -incompressible splitting surface for Y , it follows from Lemma 4.1 that P_K^Y is parallel to Q_K^A , i.e. Q_K^A is isotopic to both P_K^X and P_K^Y . In particular P_K can be properly isotoped to lie in either X_K or Y_K . By Lemma 3.7 this implies that Q_K^B must be c -incompressible in B_K , for if Q_K^B has a c -disk lying in X_K (say) we could isotope P_K to lie in X_K and be disjoint from this c -disk. By Corollary 4.2 this implies that Q_K^B is parallel to one of P_K^X or P_K^Y . As Q_K^A is parallel to both P_K^X and P_K^Y we conclude that P_K and Q_K are properly isotopic.

Case 2: Q_K^A is a single P_K^X -parallel annulus.

We will show, by contradiction, that this case does not arise. In this situation P_K^X is a single annulus and P_K^Y consists of two twice-punctured disks. See Figure 4. Recall that we have already shown that Q_K^B is c -incompressible in Y_K and P_K^Y is c -incompressible in B_K .

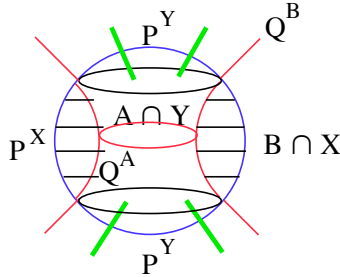


FIGURE 4

Suppose (towards a contradiction) that P_K^Y is c -incompressible in Y . By Lemma 3.6 it must be boundary compressible. As in the previous case the ∂ -compressing

disk is incident to $Q_K - P_K$ in an essential arc, i.e. one of Q_K^A or Q_K^B is ∂ -compressible to P_K^Y . The annulus Q_K^A can't ∂ -compresses to P_K^Y , since its boundary components are on different components of P_K^Y . On the other hand, if a component of Q_K^B ∂ -compresses to P_K^Y , by Corollary 4.2 it follows that Q_K^B has a twice punctured disk component parallel to one of the two components of P_K^Y . In this case $|P_K \cap Q_K|$ can be decreased by 1, and this contradicts the minimality assumption. We conclude that P_K^Y must be c-compressible in the complement of Q_K .

Suppose D^* is a c-disk for P_K^Y in the complement of Q_K . Necessarily $D^* \subset A_K$ and we may as well take ∂D^* to be one of the circles $P \cap Q$. Then D^* is also a c-disk for Q_K^A lying in $A \cap Y$. By c-weak incompressibility of Q_K any c-disks for Q_K^B would have to lie in Y_K . But we established in the beginning of this case that this is not possible so Q_K^B is in fact c-incompressible. By Lemma 3.6 Q_K^B must be boundary compressible. As we already saw, if the boundary compression is to P_K^Y , the intersection $P_K \cap Q_K$ can be reduced, so Q_K^B must be boundary compressible to the annulus P_K^X . It follows then from Lemma 4.1 that Q_K^B , like Q_K^A , is an annulus parallel to P_K^X . Then Q_K is a torus that is disjoint from K and so it cannot be a bridge surface, a contradiction. \square

Corollary 4.4. *Suppose K is a knot in S^3 , 2-bridge with respect to the bridge surface $P \cong S^2$, and K is not the unknot. Suppose Q is any other bridge surface for K . Then either*

- Q is stabilized
- Q is meridionally stabilized
- Q is perturbed or
- Q is properly isotopic to P .

Proof. If Q is c-weakly incompressible then Theorem 4.3 shows that Q is properly isotopic to P . If Q is c-strongly compressible, Theorem 3.9 says that either Q is stabilized, meridionally stabilized or perturbed, or K is removable with respect to the bridge surface Q , or there is a surface F transverse to K so that F_K is essential in S_K^3 . The last possibility does not occur for 2-bridge knots (see [HT]). Corollary 3.5 shows that if K is removable with respect to Q , then Q is stabilized. \square

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