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Thin position for 3-manifolds

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ABSTRACT. We define thin position for 3-manifolds, and examine its relation to Heegaard genus and essential surfaces in the manifold. We show that if the width of a manifold is smaller than its Heegaard genus then the manifold contains an essential surface of genus less than the Heegaard genus.

The concept of thin position for knots in the 3-sphere was developed by D. Gabai [2]. Its relation to bridge position for knots and to incompressible meridinal planar surfaces in the knot complement is discussed in [3]. We define an analogous notion of thin position for 3-manifolds, and discuss its relation to Heegaard splittings, Heegaard genus, and incompressible surfaces.

For illustrative purposes we will restrict to connected orientable closed manifolds. In principle the arguments here could be extended not only to non-orientable closed manifolds, but in fact to all compact 3-manifolds, merely by regarding a compact 3-manifold as a cobordism from one (possibly empty) collection of

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boundary components to the collection of remaining components. More elaborate generalizations are also possible, in which the sequential ordering of the intermediate surfaces, analogous to the ordering of vertices in a linear graph, is replaced by a more complicated configuration, analogous to vertices in a more complicated graph.

Any closed orientable 3-manifold M can be constructed as follows: begin with some 0-handles, add some 1-handles, then some 2-handles, then some more 1-handles, then some more 2-handles, etc., and conclude by adding some 3-handles. Of course M can be built less elaborately: in the previous description, all the 1-handles can be added at once, followed by all the 2-handles. This corresponds to a Heegaard splitting of the manifold; the 0- and 1-handles comprise one handlebody of the Heegaard splitting, the 2- and 3-handles the other. The idea of thin position is to build the manifold as first described, with a succession of 1-handles and 2-handles chosen to keep the boundaries of the intermediate steps as simple as possible.

DEFINITIONS: For M a connected, closed, orientable 3-manifold, let $M = b_0 \cup N_1 \cup T_1 \cup N_2 \cup T_2 \cup \ldots \cup N_k \cup T_k \cup b_3$, where b_0 is a collection of 0-handles, b_3 is a collection of 3-handles, and for each i, N_i is a collection of 1-handles and T_i is a collection of 2-handles. We think of building M in steps, starting with b_0, then adding N_1, then T_1, etc.

Let S_i, $1 \leq i \leq k$, be the surface obtained from $\partial[b_0 \cup N_1 \cup T_1 \cup N_2 \cup T_2 \cup \ldots \cup N_i]$ by deleting all spheres bounding 0- or 3-handles in the decomposition. Let F_i, $1 \leq i \leq k-1$, be the surface obtained from $\partial[b_0 \cup N_1 \cup T_1 \cup N_2 \cup T_2 \cup \ldots \cup T_i]$ by similarly deleting all such spheres.

Let W_i = (collar of F_{i-1}) $\cup N_i \cup T_i$ together with every 0- and 3-handle incident to N_i or T_i. W_i is divided by a copy of S_i into two compression bodies: $\overline{N_i}$ = (0-handles) $\cup$ (collar of F_{i-1}) $\cup N_i$ and $\overline{T_i}$ = (collar of S_i) $\cup T_i \cup$ (3-handles). Thus S_i describes a Heegaard splitting of W_i into compression bodies $\overline{N_i}$ and $\overline{T_i}$.

Our goal will be to find a decomposition for which the S_i are as simple as possible. To that end, for S a closed connected orientable surface other than the sphere, define the complexity $c(S) = 1 - \chi(S) = 2\text{genus}(S) - 1$. Define $c(S^2) = 0$. For S not necessarily connected define $c(S) = \sum\{c(S') \mid S' \text{ a component of } S\}$. The complexity is then always decreased by 2-surgery along an essential circle and is unchanged by 2-surgery along an inessential circle.

Define the <u>width of the decomposition</u> of M to be the set of integers $\{c(S_i) \mid 1 \leq i \leq k\}$. There is a natural way to order finite multi-sets of integers: arrange each multi-set of integers in monotonically non-increasing order, then compare the ordered multi-sets lexicographically. For example the multi-set {3, 3, 5, 3, 2, 1} is less than the multi-set {2, 2, 5, 3, 4} since [5, 3, 3, 3, 2, 1] precedes [5, 4, 3, 2, 2] in lexicographic order. Define the <u>width</u> w(M) of M to be the minimal width over all decompositions, using the above ordering on finite multi-sets of integers. A given decomposition is <u>thin</u> if the width of the decomposition is the width of M.

EXAMPLE: Suppose M is the connected sum of two lens spaces. Then M has a Heegaard splitting of genus two. The width of this decomposition is {c(genus two surface)} = {3}. But the decomposition can be re-arranged to a thin decomposition: (0-handle) ∪ (1-handle) ∪ (2-handle) ∪ (1-handle) ∪ (2-handle) ∪ (3-handle), which has width {1, 1}. This is then the width of M.

NOTE: It is straightforward to show in general that width is "additive" in the obvious manner under the connect-sum operation: $w(M_1 \# M_2) = w(M_1) \cup w(M_2)$.

There are some obvious ways in which a decomposition of M might be thinned: If a component of an F_i is an inessential sphere, then the ball it bounds contains some 1- and 2-handles, since we removed from F_i any sphere which merely bounds a 0- or 3-handle. Replace

the ball with either a 0- or a 3-handle, depending on which side of F_i the ball lies on. This will thin the decomposition. We conclude:

RULE 1: *In a thin decomposition, any sphere component of any F_i is essential.*

If there is a component of an F_{i-1} to which no 1-handle is attached, but some 2-handles in T_i are (so the component persists into S_i), the decomposition would be thinner if those 2-handles were considered to be part of T_{i-1}. Similarly, if there is a component of an F_{i-1} to which 1-handles in N_i are attached, but no 2-handles in T_i, the decomposition would be thinner if those 1-handles were regarded as part of N_{i+1} instead. We conclude:

RULE 2: *In a thin decomposition, each component of F_{i-1} either persists into F_i, or has handles from both N_i and T_i attached to it.*

There is a less obvious way to thin a decomposition: Recall that S_i describes a Heegaard splitting of W_i. This Heegaard splitting is called <u>weakly reducible</u> [1] if there are essential disks D_N in $\overline{N_i}$ and D_T in $\overline{T_i}$ such that $\partial D_N \cap \partial D_T = \emptyset$ in S_i. In particular, if W_i contains a 1-handle in one component and a 2-handle in another, then W_i is automatically weakly reducible.

RULE 3: *In a thin decomposition, no W_i is weakly reducible.*

PROOF: Suppose W_i is weakly reducible. Remove a neighborhood of D_N from $\overline{N}$, converting it into a compression body N' with one fewer 1-handle or one more component. Then attach a 2-handle with core D_T to N'. Next attach the 1-handle dual to D_N, followed by the remainder of the 2-handles. This replaces the original decomposition in W_i, which yields the integer $\{c(S_i)\}$, by a decomposition with two surfaces S_{i_-} and S_{i_+}, obtained from S_i by compressing along ∂D_N and ∂D_T respectively. Since $c(S_{i_\pm}) < c(S_i)$ the new decomposition is thinner.

As an immediate corollary we have:

RULE 4: *In a thin decomposition, all handles in N_i and T_i are incident to the same component of S_i, called the* <u>*active*</u> *component of S_i.*

PROOF: Otherwise W_i would contain two components each with both 1- and 2-handles, so W_i would be weakly reducible.

EXAMPLE: If S is the splitting surface of a weakly reducible Heegaard splitting of M, then width(M) < $\{c(S)\}$.

LEMMA: *If $\partial W_i = F_{i-1} \cup F_i$ is compressible in W_i, then W_i is weakly reducible.*

PROOF: See [1].

RULE 5: *In a thin decomposition, each component of each F_i is incompressible in M.*

PROOF: Let D be a compressing disk for a component of F_i. Let $F = \cup F_i$. By an innermost disk argument on $D \cap F$ we can assume $D \cap F = \partial D \subset F_i$, so D lies entirely inside either W_i or W_{i+1}, say the former. The lemma implies that W_i is weakly reducible, contradicting Rule 3.

DEFINITION: A separating surface S in a 3-manifold M is <u>weakly incompressible</u> (see [3]) if any two compressing disks for S on opposite sides of S intersect along their boundary.

RULE 6: *In a thin decomposition of a 3-manifold, each surface S_i is weakly incompressible.*

PROOF: By rule 5, F is incompressible, so we can

assume any compressing disk for S_i lies in W_i. Hence if S_i is not weakly incompressible, W_i is weakly reducible, contradicting rule 3.

RULE 7: *If M is irreducible and not a Lens space, then in a thin decomposition no component of any S_i is a torus. In particular no number in w(M) is smaller than 3.*

PROOF: Suppose C is a torus component of some S_i. The decomposition can't be a Heegaard splitting, since M is not a Lens space. So assume that $k > 1$ and, with no loss of generality, assume that C is the active component of S_i. Let W be the component of W_i containing C. Then $\partial W \subset F$ is a non-empty (since S_i is not a Heegaard splitting) collection of spheres. Rule 1 says that such a sphere component of F would be essential, which is impossible since M is irreducible.

COROLLARY: *Let g be the Heegaard genus of M. If M is irreducible and contains no incompressible surfaces of genus < g, then a minimal genus Heegaard splitting for M is a thin decomposition. Hence $w(M) = \{2g - 1\}$.*

In particular, this is true if M is irreducible and non-Haken.

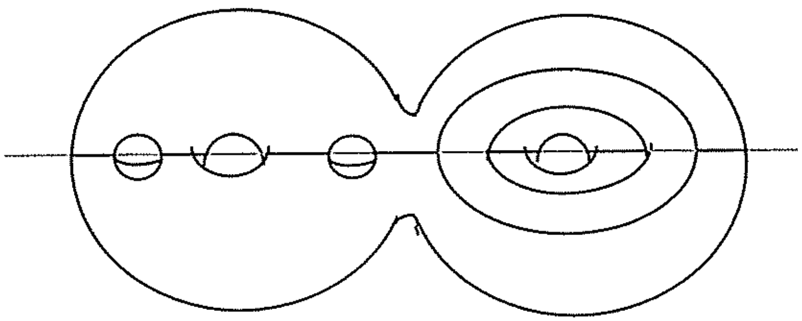
COROLLARY: *Let Σ be a homotopy 3-sphere. If $w(\Sigma) < \{5\}$ then $\Sigma = S^3$.*

PROOF: We may as well take Σ to be irreducible, and it clearly contains no incompressible surfaces, so if g is the Heegaard genus of Σ, $w(\Sigma) = \{2g-1\}$. If $2g - 1 < 5$, then $g \leq 2$, which forces Σ to be standard.
More generally:

COROLLARY: *If $w(M) < \{5\}$ then M is a 2-fold cover of $\#(S^1 \times S^2)$ branched along a link.*

PROOF: Choose a thin decomposition of M and let S =

$\cup S_i$ and $F = \cup F_i$. Then each component of $M - (S \cup F)$ is a (possibly punctured) compression body H with genus$(\partial_+ H) \leq 2$. On each such compression body there is an obvious involution (rotation by π about the illustrated axis) with 1-dimensional fixed point set. The quotient is a possibly punctured 3-ball. It's well-known that this involution commutes with the mapping class group of the boundary surfaces, so it extends across the gluing maps when the compression bodies are glued together along S and F. The quotient of M is obtained from punctured 3-balls by identifying boundary components, so it's a possibly trivial ($= S^3$) connected sum of $(S^1 \times S^2)$'s.



EXAMPLE: Let Q be a genus $g > 0$ orientable surface and $M = Q \times S^1$. We will show that M has width $\{3, 3, \ldots, 3\}$, with $2g$ copies of 3.

Step 1: A decomposition of this width exists: First observe that (annulus) x S^1 and (pair of pants) $\times S^1$ each admit a genus 2 Heegaard decomposition in which precisely two boundary components lie on one of the compression bodies. But Q can be constructed by successively attaching $2g - 2$ pairs of pants to an annulus, and then topping off with a final annulus. Upon crossing this picture with S^1, the Heegaard splittings of each piece amalgamate to give the required decomposition of (surface) x S^1.

Step 2: We show that no thinner decomposition exists.

Let $w(M)$ be the set of integers from a thin decomposition. It follows from rule 7 and step 1 that the only integer in $w(M)$ is 3 - say $w(M)$ consists of r copies of 3. Again by rule 7 each S_i, $1 \leq i \leq r$ must then be a single genus two surface. It follows that each W_i' is connected and that $b_0 \cup N_1$ is a genus two handlebody. No F_i can be a sphere, so each W_i, $1 < i \leq r$ contains a single 1-handle. Hence M can be constructed with $(r+1)$ 1-handles. But $H_1(M)$ has rank $2g + 1$, so we conclude $r \geq 2g$, as required.

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