A STRONG HAKEN'S THEOREM

MARTIN SCHARLEMANN

Abstract. Suppose $M = A \cup_T B$ is a Heegaard split compact orientable 3-manifold and $S \subset M$ is a reducing sphere for M. Haken [Ha] showed that there is then also a reducing sphere S^* for the Heegaard splitting. Casson-Gordon [CG] extended the result to ∂-reducing disks in M and noted that in both cases S^* is obtained from S by a sequence of operations called 1-surgeries. Here we show that in fact one may take $S^* = S$, at least in the case where M contains no $S^1 \times S^2$ summands.

It is a foundational theorem of Haken [Ha] that any Heegaard splitting $M = A \cup_T B$ of a closed orientable reducible 3-manifold M is reducible; that is, there is an essential sphere in the manifold that intersects T in a single circle. Casson-Gordon [CG, Lemma 1.1] refined and generalized the theorem, showing that it applies also to essential disks, when M has boundary. More specifically, if S is a disjoint union of essential disks and 2-spheres in M then there is a similar family S^*, obtained from S by ambient 1-surgery and isotopy, so that each component of S^* intersects T in a single circle. In particular, if M is irreducible, so S consists entirely of disks, S^* is isotopic to S.

There is of course a more natural statement, in which S does not have to be replaced by S^*, but a proof of the natural statement has been elusive. Here we present such a proof, in the case that M contains no $S^1 \times S^2$ summands. The requirement that every sphere in M is separating is used frequently in the proof, but perhaps can be circumvented. A reader who would like to get the main idea in a short amount of time could start with the example in Section 8.

1. Introduction and Review

Suppose T is a Heegaard surface for a compact orientable 3-manifold $M = A \cup_T B$ and D is a ∂ reducing disk for M, with $\partial D \subset \partial_- B \subset \partial M$. Recall:

Theorem 1.1 (Haken, Casson-Gordon)**.** *There is a ∂-reducing disk E for M such that*

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- *$\partial E = \partial D$*
- *E intersects T in a single essential circle (i. e. E ∂-reduces T)*

Note: D and E are isotopic if M is irreducible; but if M is reducible then there is no claim that D and E are isotopic.

Similarly, if M is reducible, there is a reducing sphere for M that intersects T in a single circle (i. e. it is a reducing sphere for T). But Haken made no claim that the reducing sphere for T is isotopic to a given reducing sphere for M.

Theorem 1.2 (Strong Haken)**.** *Suppose S is a properly embedded surface in M so that each component of S is either a ∂-reducing disk or a reducing sphere in M. Suppose also that each 2-sphere in M is separating. Then there is an isotopy of T so that afterwards each component of S is either a ∂-reducing disk or reducing sphere for T.*

A reducing sphere (or a properly embedded essential disk) could lie entirely in A (or B) if it cuts off a punctured 3-ball from A. Such a component could easily be isotoped to intersect T in a single circle just by poking a disk of it into B. On these grounds we henceforth ignore (that is, remove from S) any components that can be isotoped to lie entirely in A or B.

We will further assume in the proof that $\partial S \subset \partial_- B \subset \partial M$. The case in which S contains ∂-reducing disks with boundaries in both $\partial_- A$ and $\partial_- B$ follows, with little more than this addendum: First ignore those disks in S whose boundaries lie in $\partial_- A$ and apply the proof below to the remainder of S. Then ∂-reduce (M, T) along all the disk components of S incident to $\partial_- B$, and apply the proof again with the roles of A and B switched.

Let Σ denote a spine of B, that is (a thin regular neighborhood of) the union of $\partial_- B$ and a certain type of graph in B: all valence 1 vertices in the graph lie on $\partial_- B$, all other vertices have valence 3, and B deformation retracts to Σ.

The deformation retraction of B to Σ will carry a disjoint collection Δ of meridians of A to disks in $M - \Sigma$; continue to denote these Δ. Such a collection is *complete* if the complement of $B_+ = B \cup \eta(\Delta)$ is the union of balls and a collar of $\partial_- A \subset \partial M$. Suppose some edge e of Σ is disjoint from such a complete collection. A point on e corresponds to a meridian of B whose boundary lies on ∂B_+. If it is inessential in ∂B_+ then it bounds a disk in A, so such a meridian can be completed to a reducing sphere for T. We call such an edge a *reducing edge* of Σ. Since every sphere separates M, any reducing edge separates Σ.

The other possibility is that the boundary of the meridian disk for e is essential on ∂B_+, so it, together with an essential curve in $\partial_- A$ bounds an annulus $a_e \subset A$. Together, the meridian disk of e and the annulus a_e comprise a boundary reducing disk for M, in fact one that also ∂-reduces the splitting surface T. We will eliminate from consideration this possibility by a straightforward trick, which we now describe.

Lemma 1.3. *There is a collection $C \subset \partial_- A$ of disjoint simple closed curves with the property that C essentially intersects any simple closed curve in $\partial_- A$ that bounds a disk in M.*

Proof. Suppose A_0 is a genus g component of $\partial_- A$. By standard duality arguments, the collection $K \subset A_0$ of simple closed curves that compress in M can generate at most a g-dimensional subspace of $H_1(A_0, \mathbb{R}) \cong \mathbb{R}^{2g}$. More specifically, one can find a non-separating collection $c_1, ...c_g$ of disjoint simple closed curves in A_0 so that $C_- = \cup_{i=1}^{g} c_i$ generates a complementary g-dimensional subspace of $H_1(A_0, \mathbb{R})$, and therefore essentially intersects any *non-separating* curve in K. It is easy to add to C_- a further disjoint collection of $2g - 3$ simple closed curves, each non-separating, so that the result $C_0 \subset A_0$ has complement a collection of $2g - 2$ pairs of pants. Since each curve in C_0 is non-separating, and any simple closed curve in the pairs of pants $A_0 - C_0$ must be parallel to a curve in C_0, any curve in K essentially intersects C_0.

Do the same in each component of $\partial_- A$; the result is the required collection C. □

Following Lemma 1.3 add to the collection of disks Δ the annuli $C \times I \subset \partial_- A \times I \subset M - B_+$, and continue to call the collection of disks and annuli Δ. Then a meridian of an edge e of Σ that is disjoint from the (newly augmented) Δ cannot be part of a ∂-reducing disk for T and so must be part of a reducing sphere. Since the collection S of reducing spheres and ∂-reducing disks we are considering have no contact with $\partial_- A$, arcs of $S \cap \Delta$ are nowhere incident to $\partial_- A$. Hence the annuli which we have added to Δ can be viewed as once-punctured disks, and the arguments cited below, usually applied to disk components of Δ apply also to the newly added annuli components.

2. Reducing edges and S

Lemma 2.1. *Suppose a spine Σ for B and a complete collection of meridians (and annuli as above) Δ for A have been chosen to minimize the pair $(|\Sigma \cap S|, |\partial\Delta \cap S|)$ (lexicographically ordered, with Σ, S, Δ all in general position). Then Σ intersects S only in reducing edges.*

Notes:

- We do not care about the number of circles in $\Delta \cap S$.
- If any disk component of S intersects Σ only in $\partial S \subset \partial_- B$, it is a ∂-reducing disk for T
- If any sphere component of S intersects Σ in a single point, then it is a reducing sphere for T.

Proof. Recall from a standard proof of Haken's Theorem (see eg [Sc], [ST]) that $(\Sigma \cup \Delta) \cap S$ (ignoring circles of intersection) can be viewed as a graph Γ in S in which points of $\Sigma \cap S$ are the vertices and $\Delta \cap S$ are the edges. With slight abuse of notation, we will also regard boundary components of S as vertices in the graph, since they lie in $\partial_- B \subset \Sigma$. This can be made sensible by imagining capping off each boundary component of S by an imaginary disk outside M.

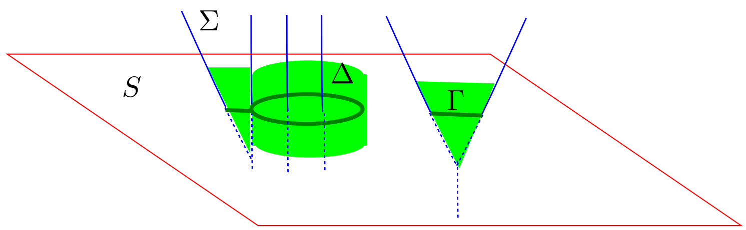

FIGURE 1. The graph Γ

There must be at least one vertex in each reducing sphere of S since we have deleted from S any component that lies entirely in A. Similarly, there is at least one "vertex" in each ∂-reducing disk in S, namely the disk's boundary.

A loop in Γ is essential if there are vertices in the the disk it cuts off from S. The classical proof shows that inessential loops can be removed by altering Δ and thereby reducing $|\partial\Delta \cap S|$ by 2.

That $|\Sigma \cap S|$ is minimal then guarantees

- any innermost loop in Γ contains only isolated vertices
- if there are no loops in Γ then every vertex is isolated.

It follows that there is at least one isolated vertex, and this means that its edge in Σ is reducing. This establishes the original Haken's Theorem and, if there are no loops at all, further establishes Lemma 2.1. That there are no loops is what we now show.

Consider an innermost loop, consisting of a vertex $p \in \Sigma \cap S$ and an edge (the loop) lying in a component D of Δ. Together, they define a circle c in S lying in the boundary of the 3-manifold $A_- = A - \eta(D)$.

The circle c bounds a disk E in S which, by the classic argument noted above, intersects Σ only in reducing edges. It follows immediately that c is null-homotopic in A_- and then by Dehn's lemma that it bounds an embedded disk E' entirely in A_-.

By standard innermost disk arguments we can assume the interior of E' is disjoint from Δ. Now split D in two by compressing the loop to the vertex along E' and replace D in Δ by these two pieces, creating a new complete collection of disks and annuli Δ', with $|\partial\Delta' \cap S| \leq |\partial\Delta \cap S| - 2$. Since we have introduced no new vertices, this contradicts our assumption that $(|\Sigma \cap S|, |\partial\Delta \cap S|)$ is minimal. $\square$

Note that the new Δ' may intersect S in many more circles than Δ did, but we don't care.

3. Reducing spheres for Σ

Arbitrarily pick a point $*$ in ∂M (or just in M if M is closed) and call it an outermost point. For R a sphere in M and p a point in $M - R$ we say p is *inside* (resp *outside*) R if a path from $*$ to p intersects R in an odd (resp even) number of points. Since R separates M, this is well-defined.

Given a reducing edge e in Σ the associated reducing sphere R_e for e is the reducing sphere for T that passes once through e, at a point very near the end of e that lies inside R_e. It's easy to see that any other reducing sphere passing once through e is isotopic to R_e in M via reducing spheres, so R_e is well-defined.

For e and R_e as above, call $e \cup R_e$ a *flower*, with e the stem and R_e the *blossom*. The point $e \cap R_e$ is the *base* of the blossom, and the other end of e is the base of both the stem and the flower. Note that if any blossom is entirely disjoint from S then we may as well reduce M along it and separate M into two components, and proceed with a proof of the Strong Haken Theorem in each. So if we detect a blossom that is disjoint from S, (call the blossom *inert*), then we are done. So henceforth we assume that no blossom is inert.

Given a spine Σ, the collection of reducing spheres for T associated to reducing edges of Σ is called the set of reducing spheres associated with Σ. (There are many more reducing spheres for T; they intersect the neighborhood of the spine in disks that are not meridians.) We will be interested in the subset, denoted $\mathfrak{R}$, that are associated to edges that intersect S.

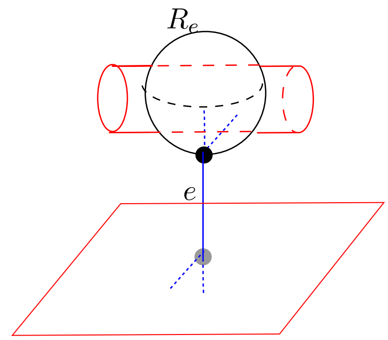

FIGURE 2. Flower and its two base points

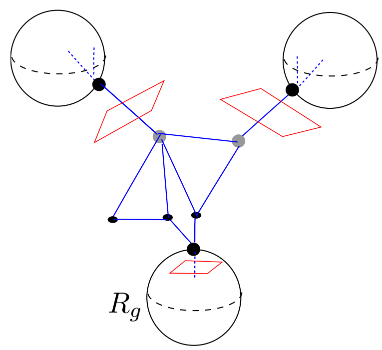

FIGURE 3. $M_{\mathfrak{R}}$

Let $M_{\mathfrak{R}}$ be a component of $M - \mathfrak{R}$ and consider those elements $\mathfrak{R}_0$ that are incident to it. Among the components of $\mathfrak{R}_0$ there is exactly one, denoted R_g, that lies outside all the others. Call it the *ground* component of $\mathfrak{R}_0$. The stem of R_g then lies outside of $M_{\mathfrak{R}}$ and the stems of all other elements of $\mathfrak{R}_0$ lie inside $M_{\mathfrak{R}}$. The two edges of Σ that are incident to R_g and lie inside $M_{\mathfrak{R}}$ are called branches (of Σ in $M_{\mathfrak{R}}$).

4. STEM SLIDES

Lemma 4.1. *Suppose C is a compression-body with $p \in \partial_+ C$ and $q \in interior(C)$. Suppose α, β are two arcs from p to q in C. Then, perhaps first sliding the end of β at p around a closed path in $\partial_+ C$ and then allowing points of the arc β to pass through the arc α, β can be isotoped rel endpoints to α in C.*

Proof. Let Σ be a spine for the compression-body C. By general position, we may take Σ to be disjoint from the knot $K = \alpha \cup \beta$. K represents an element of $\pi_1(C)$; since $\pi_1(\partial_+ C) \to \pi_1(C)$ is surjective

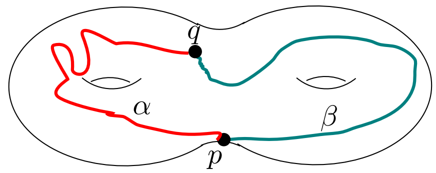

FIGURE 4

we can slide the end of β around a closed path in $\partial_+ C$ so that afterwards K is null-homotopic in C. Now apply [FS, Theorem 0] to the "lollypop" K, which lies in the collar $C - \eta(\Sigma)$ of $\partial_+ C$. According to [FS, Theorem 0] the knot K can be altered, rel p, by crossing changes, until it bounds a disk Δ in C. By appropriately shrinking or lengthening α and β one can guarantee that each such crossing change passes the arc β through α, so it is allowed by our hypothesis. Now use Δ to isotope α to β rel end points. □

5. STEM SWAPS

Proposition 5.1 (Stem Swapping)**.** *Let σ be a stem with blossom $\mathfrak{a}$ and $\sigma' \subset M - \Sigma$ be a disjoint arc with one end at the base of $\mathfrak{a}$ and the other at a point in an edge of Σ. Then the complex Σ' obtained from Σ by replacing the stem σ with the stem σ' is also a spine for T. That is, T is isotopic to a regular neighborhood of Σ'.*

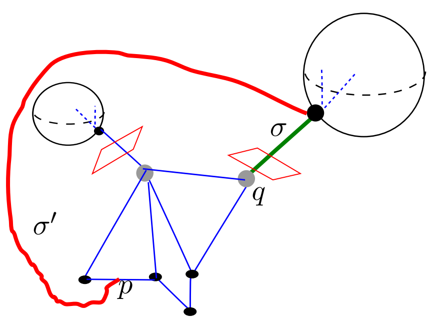

FIGURE 5. A stem swap

Proof. Let γ be a path in Σ whose endpoints are the base points of the stems σ and σ'. Apply Lemma 4.1 to the arcs $\beta = \sigma \cup \gamma$ and $\alpha = \sigma'$. □

This operation is called a *stem swap*. A refinement of the above will be useful; its proof is identical.

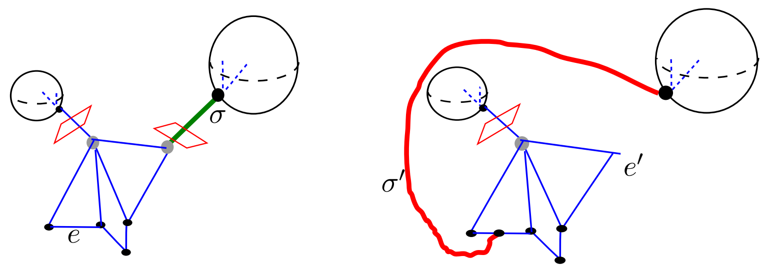

FIGURE 6. Spines Σ and Σ'

Corollary 5.2. *Let $M_{\mathfrak{R}}$ be a component of $M - \mathfrak{R}$. If σ and σ' both lie in $M_{\mathfrak{R}}$, then the isotopy of T described in Proposition 5.1 takes place entirely in $M_{\mathfrak{R}}$.* □

The only difference between the two spines Σ and Σ' (other than the obvious switch from σ to σ') is that the edge $e \subset \Sigma$ that contains p becomes two edges in Σ' and the edge $e' \subset \Sigma'$ that contains q was actually two edges in Σ.

Let $\mathfrak{R}$ and $\mathfrak{R}'$ be the set of reducing spheres associated with respectively the spines Σ and Σ' in Proposition 5.1. Suppose σ and σ' both lie in the same component $M_{\mathfrak{R}}$ of $M - \mathfrak{R}$. The base p of σ' divides e into two edges of Σ'. In Lemma 5.3 below, let e_o denote the piece of e that lies on the outer side of p, if e is separating. If e is non-separating, either piece could be e_o.

Lemma 5.3. *If e_o does not intersect S then $\mathfrak{R}' \subset \mathfrak{R}$.*

Note that if σ' is disjoint from S then $\mathfrak{a} \notin \mathfrak{R}'$ so $\mathfrak{R}'$ and $\mathfrak{R}$ are not necessarily equal.

Proof. Let F be a flower whose blossom $\mathfrak{f}$ is in $\mathfrak{R}'$. Our goal is to show that $\mathfrak{f}$ is in $\mathfrak{R}$.

We first show that σ can be isotoped off of $\mathfrak{f}$. The points of $\mathfrak{f} \cap \sigma$, can successively be pushed along σ and across the blossom $\mathfrak{a}$ until the resulting sphere $\mathfrak{f}'$ is disjoint from σ. Then the only point in which $\mathfrak{f}'$ intersects Σ is at its base, where it is attached to the stem ρ of the flower F. In particular the union of ρ and $\mathfrak{f}'$ is a flower, and since we are given that ρ intersects S, the blossom $\mathfrak{f}'$ is in $\mathfrak{R}$ or, more precisely, is parallel to an element of $\mathfrak{R}$. In particular, any part of σ' that intersects the product region between $\mathfrak{f}'$ and an element of $\mathfrak{R}$ can be isotoped off of $\mathfrak{f}'$. Reversing the process, it follows that any part of σ passing through $\mathfrak{f}$ can be isotoped off of it.

FIgure 7 makes this clear: Visualize $\sigma \cup \sigma'$ as a properly imbedded arc in A, on which $\mathfrak{a}$ sits as a bead. The process described above simply

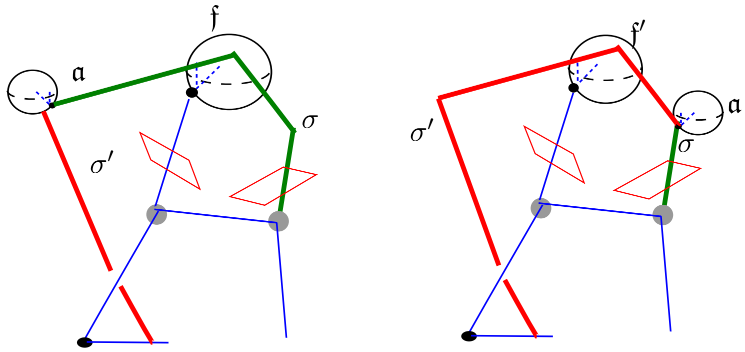

FIGURE 7. Blossoms $\mathfrak{f}$ and $\mathfrak{f}'$

moves the bead back and forth along the arc through $\mathfrak{f}$. From this point of view isotoping σ' off of $\mathfrak{f}'$, will automatically isotope σ off of $\mathfrak{f}$.

Once σ is disjoint from $\mathfrak{f}$ we only need to show that the edge of Σ that is incident to $\mathfrak{f}$ (which is the stem for $\mathfrak{f}$ in Σ) intersects S outside of the sphere $\mathfrak{f}$. We consider the possibilities:

If $\mathfrak{f}$ is disjoint from both e and e' then the stem of F in Σ' is also an edge of Σ and we are done.

If $\mathfrak{f}$ crosses through e, then e is separating. Then the proviso on p in the hypothesis ensures that the stem of F does not intersect S, so it could not have been in $\mathfrak{R}'$.

So assume that $e \neq e'$ and e' is the stem of F. This implies that $\mathfrak{f}$ intersects e' at its inside end and S also intersects e'.

The base point q divides e' into an inside edge e'_i and an outside edge e'_o. The former edge e'_i is the stem for $\mathfrak{f}$ in Σ, so if S intersects e'_i then we are done. So we may as well assume S intersects (only) e'_o. Tubing together $\mathfrak{f}$ and $\mathfrak{a}$ along their stems then exhibits a sphere R_1 whose stem is e'_o. (See Figure 8.) So the (apparently new) sphere R_1 actually lies in $\mathfrak{R}$, with stem e'_o and branches σ and e'_i . The arc σ' can't penetrate F or R_1, so there is no edge on which p can lie, a contradiction. □

6. MINIMIZING $\mathfrak{R} \cap S$

Consider all spines that intersect S only in reducing edges. To each such spine Σ let $\mathfrak{R}(\Sigma)$ denote the corresponding collection of reducing spheres for Σ whose stem intersects S. Among all such spines choose that which minimizes the number of (circle) components of $\mathfrak{R}(\Sigma) \cap S$. Then

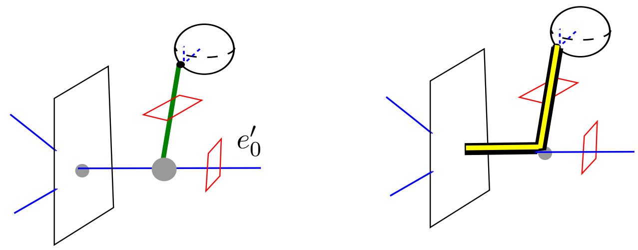

FIGURE 8. Noticing a 'new' sphere

Proposition 6.1. $\mathfrak{R}(\Sigma)$ *is disjoint from* S.

Note that for this proposition we don't care about how often the reducing edges of the spine Σ intersects S. We revert to the notation $\mathfrak{R}$ for $\mathfrak{R}(\Sigma)$.

Proof. We wish to study disk components of $S - \mathfrak{R}$. One sort of disk that might arise is one that lies very near the base of a flower, intersecting the stem once. Such a disk and its bounding circle in $\mathfrak{R}$ will be called *fake* because it can easily be removed by moving the blossom slightly along the stem, violating only our convention that each component of $\mathfrak{R}$ lies very close to the innermost end of its stem. The disk is then said to be *recessed away.*

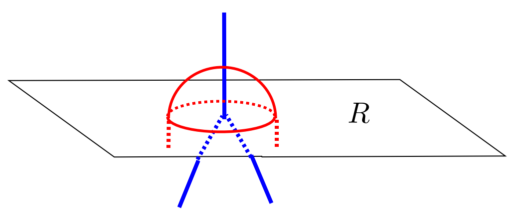

FIGURE 9. Fake disk

Among the components of $\mathfrak{R} \cap S$ pick c to be one that is innermost in S among those that are not fake. Recess away any fake disks it contains. Let $E \subset S$ be the disk that c bounds in S and let $M_{\mathfrak{R}}$ be the component of $M - \mathfrak{R}$ in which E lies. As previously, let $\mathfrak{R}_0 \subset \mathfrak{R}$ denote the collection of spheres $\mathfrak{R} \cap M_{\mathfrak{R}}$ in $\partial M_{\mathfrak{R}}$ and let $R_0 \in \mathfrak{R}_0$ be the reducing sphere on which c lies, p be the base of R_0, and $D \subset R_0$ be the disk c bounds in $R_0 - p$.

We will assume that R_0 is not the ground sphere for $M_{\mathfrak{R}}$; the proof when it is the ground sphere is slightly different and will be considered afterwards. E divides $M_{\mathfrak{R}}$ into two components X and Y; let X be the component that contains the base q of the stem for R_0.

Lemma 6.2. *X contains the base points of all stems in $M_{\mathfrak{R}}$.*

Proof. Suppose there were a stem with base point q' in Y. Since Σ is connected, there is a path in Σ from q' to q; let γ be the shortest such path. γ must remain in $M_{\mathfrak{R}}$ since each sphere in $\mathfrak{R}_0$ is separating, so in particular γ can traverse no stem (including the stem of R_0 if p lies in Y). But each point in $\Sigma \cap \partial Y$ is, by hypothesis, on a stem, so γ cannot leave Y to reach q. □

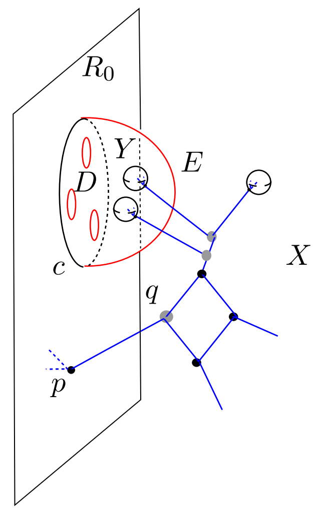

FIGURE 10

Lemma 6.3. *After perhaps some stem swaps, we can assume that each stem intersects E in at most one point.*

Proof. The figure shows how to remove multiple intersections on the same stem by a stem swap. (Note that this could completely eliminate intersections of the stem with S, removing the blossom from $\mathfrak{R}$.) □

Consider then the collection of flowers whose blossoms lie in Y and whose stems therefore intersect E exactly once. Let $\mathfrak{a}$ be the blossom of one and σ its stem. Let $M_{\mathfrak{R}}'$ be the component of $M - \mathfrak{R}$ lying on the other side of R_0. (Since R_0 is not the ground component of $\mathfrak{R}_0$, it is the ground component of $\mathfrak{R} \cap M_{\mathfrak{R}}'$.)

Choose a point of $\Sigma \cap M_{\mathfrak{R}}'$ according to the recipe given in Lemma 5.3, for example a point on a branch near R_0, and choose a path σ' in $M_{\mathfrak{R}} \cup M_{\mathfrak{R}}'$ from the point to the base of $\mathfrak{a}$. The path can be chosen so

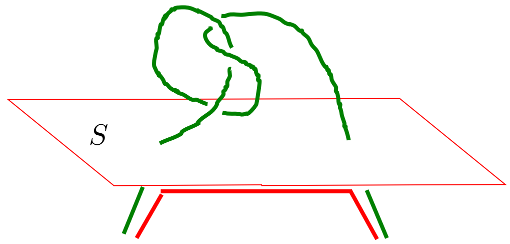

FIGURE 11

that it intersects ∂Y exactly once, but in D not in E, and is otherwise disjoint from $\mathfrak{R}$; Figure 12 shows an obvious choice.

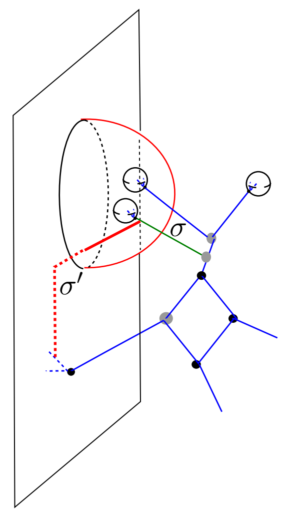

FIGURE 12

Now apply Proposition 5.1 to replace σ with σ'. Do the same operation for all blossoms in Y. Although the setting is not exactly that of Lemma 5.3 we can still apply the argument there to conclude that after the swap the only change in $\mathfrak{R}$ is that (unimportantly) some blossoms may disappear from $\mathfrak{R}$ (if their new stems avoid S) and, importantly, the reducing sphere R_0 is replaced by R_0' obtained from R_0 by replacing D with E. In any case, the number of circles $S \cap \mathfrak{R}$ drops by at least one, namely the circle $c = \partial D = \partial E$, contradicting the definition of $\mathfrak{R}$ and thereby proving the Proposition in the case when $p \in \partial X$.

Now consider the case $p \in \partial Y$. See Figure 13. If there were no blossoms in Y then E would be a fake disk, contrary to hypothesis.

If S is disjoint from the interior of the disk $D' = R_0 - D$ containing p then do a stem swap on the stems of the blossoms in Y and of the stem of R_0 as shown at the right in Figure 13; then the disk bounded by D' and E is a reducing disk for M that is disjoint from S, so it is inert, contrary to assumption.

So we assume that S intersects D'. Now apply the argument above to replace D' by E and R_0 by R'_0. This removes from $\mathfrak{R} \cap S$ all the components of $S \cap D'$, so we can ignore the fact that c is still in $\mathfrak{R}' \cap S$, but fake, and deduce that $|\mathfrak{R} \cap S|$ has been reduced, contrary to hypothesis. This concludes the proof of Proposition 6.1 when R_0 is not the ground sphere.

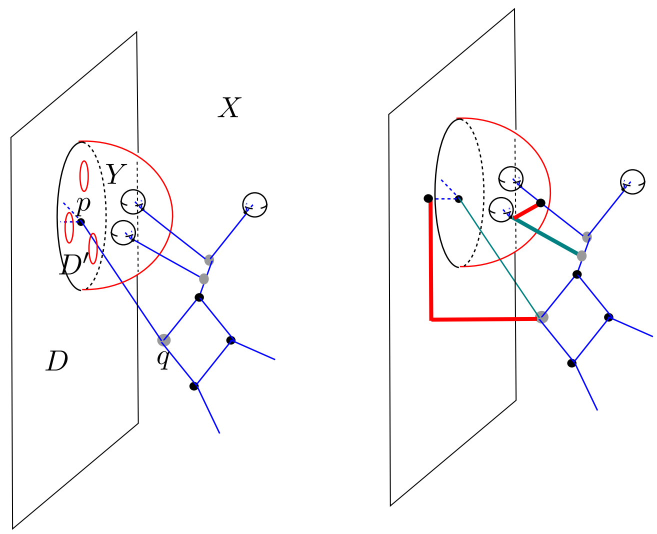

FIGURE 13

Some adjustment to the above argument is needed if c lies on the ground sphere R_g. In this case there are two edges (the branches) incident to the vertex p in R_g and neither is necessarily a stem. In this context, let X denote the component of $M_{\mathfrak{R}} - E$ that contains p and Y denote the other component.

If neither branch is a stem, essentially the same argument works, using stem swaps as shown in Figure 14.

If exactly one branch is a stem for $\mathfrak{R}$ and its blossom lies in X the same argument applies. If exactly one branch is a stem for $\mathfrak{R}$ and its blossom σ lies in Y it is tempting to do the same swap with that blossom, but there is a problem: when applied to the stem branch at p, our conventions dictate that the base of the flower R_g is then moved to the base of its stem, and that the stem itself intersects S. Thus the move may *increase* $|\mathfrak{R} \cap S|$ in a hidden way.

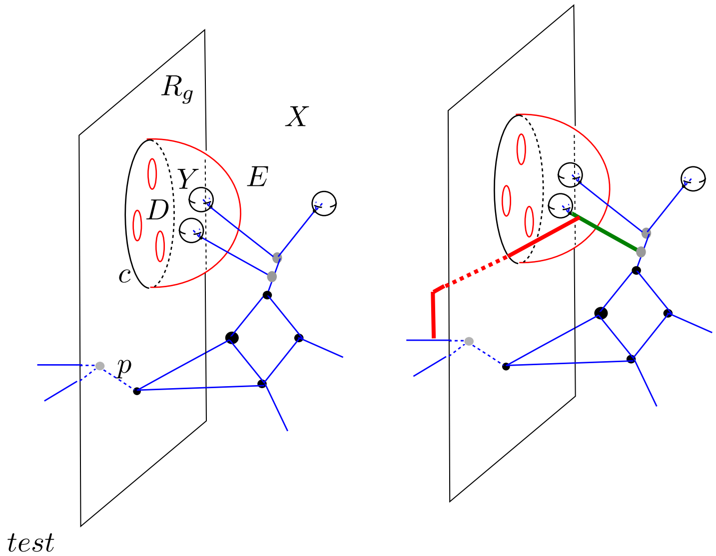

FIGURE 14

On the other hand, the non-stem branch is visibly the stem of a flower, and the fact that the blossom is not in $\mathfrak{R}$ means that the stem is disjoint from S. So a different swap, as shown in Figure 15, will move $\mathfrak{R}$ to the other end of the branch, adding it to the stem, without increasing $|\mathfrak{R} \cap S|$. It transmutes the situation to the case considered above in which neither branch is a stem.

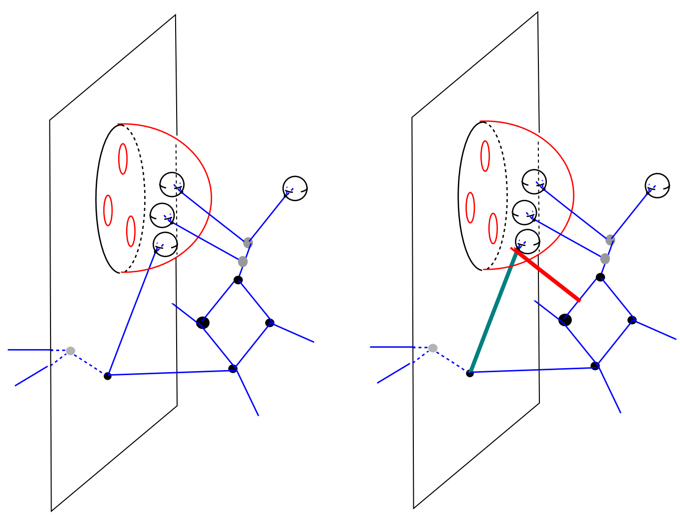

FIGURE 15

If both branches are stems and both blossoms lie in X (resp Y), E can just be isotoped through Y (resp X) and off of R_g. If both branches are stems and one blossom lies in X and the other in Y, the stem-swap shown in Figure 16 changes R_g to a different reducing sphere, as discussed above. But the new reducing sphere is parallel to the blossom that lies in X and so can be discarded completely from $\mathfrak{R}$. This reduces $|\mathfrak{R} \cap S|$ by $|R \cap S| > 1$, completing the proof of Proposition 6.1. □

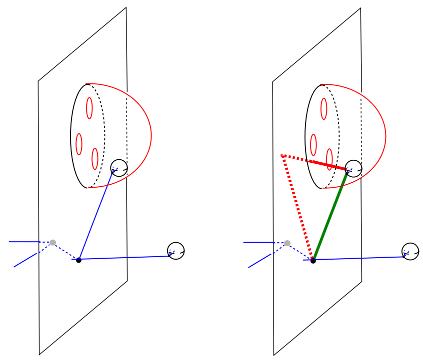

FIGURE 16

7. Conclusion

Proposition 7.1. *Suppose Σ intersects S only in reducing edges, and the associated set $\mathfrak{R}$ of reducing spheres is disjoint from Σ. Then T can be isotoped (via edge slides of Σ) so that each component of S is a reducing sphere or ∂-reducing disk for T.*

Proof. Via a stem swap as shown in Figure 17 we can reduce the number of edges intersecting S until Σ intersects S only in a single point if S is a sphere, or only in ∂S if S is a disk (shown). □

The sequence of Lemma 2.1 and Propositions 6.1 and 7.1 establishes Theorem 1.2, the Strong Haken Theorem.

8. The Zupan example

Some time ago, Alex Zupan proposed a simple example for which the Strong Haken Theorem seemed unlikely [Zu]. The initial setting is of a Heegaard split 3-manifold $M = A \cup_T B$ that is the connected sum of compact manifolds M_1, M_2, M_3 as shown in Figure 18. The blue indicates the spine Σ of B, say. The spine is not shown inside of summands M_1 and M_2 because those parts are irrelevant to the argument; psychologically it's best to think of them as spherical boundary components of M. The spine is shown for M_3 as a torus boundary component (for

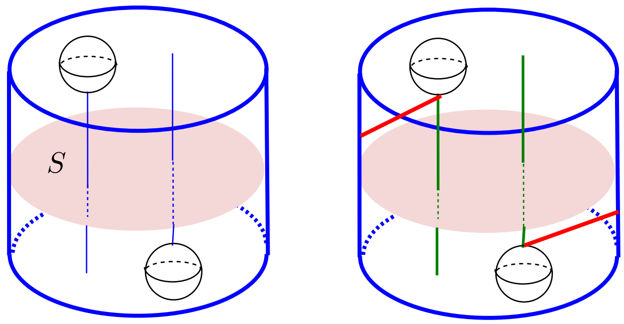

Figure 17. Swaps clearing S of final vertices

example if M_3 were a solid torus) but any Heegaard spine in any M_3 would do. An important role is played by the handlebody A that is the complement of Σ outside M_1 and M_2. In Figure 18, A is a solid torus.

We are also given a sphere S which is the tube sum of the reducing spheres for M_1 and M_2 along a tube in M_3 which can be arbitrarily complicated. This is shown in red in Figure 18. Note that passing one of the blue arcs through the red tube in Figure 18 is perfectly legitimate.

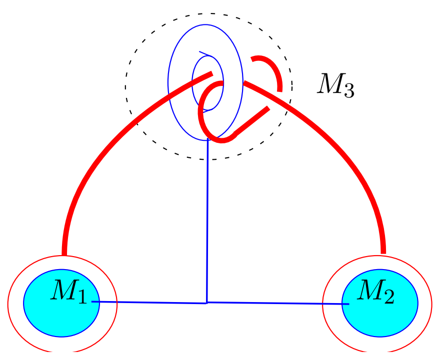

Figure 18. The initial setting

Continuing with pure psychology, Figure 19 is the same, but we have distinguished part of Σ (the rightmost edge) by turning it teal and beginning to slide it on the rest of the spine:

Now we invoke Lemma 4.1 and how its proof applies: Because $\pi_1(\partial A) \to \pi_1(A)$ is surjective, and the slides take place in ∂A, one can slide the end of the teal arc around on the rest of Σ (in fact never entering M_1 or M_2) until it is *homotopic* rel end points to the path that is the union of the tube of S and the two dotted red arcs shown in Figure 20.

And so we continue towards applying Lemma 4.1, with A playing the role of compression-body C; the reducing sphere cutting off M_2 playing

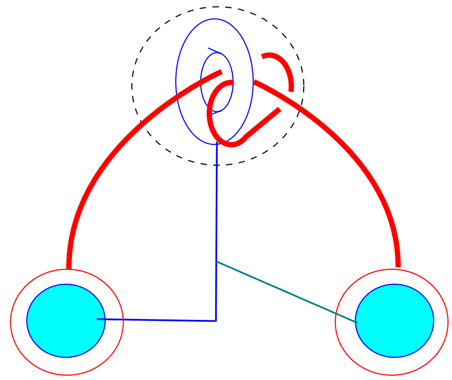

FIGURE 19. One blue edge now teal

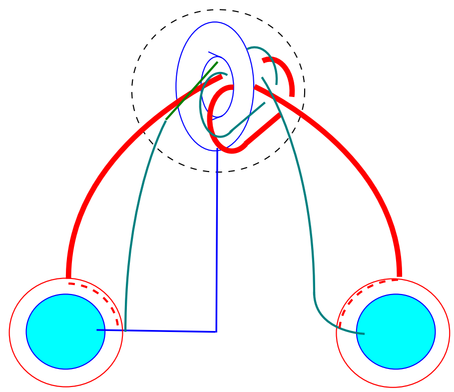

FIGURE 20. Teal edge now homotopic to red tube

the role of the point q; the other end of the teal arc playing the role of p; the teal arc playing the role of β; and the union of the red tube and the two dotted arcs in Figure 20 playing the role of α.

The result is shown in Figure 21; the teal edge now goes right through the tube, never intersecting S.

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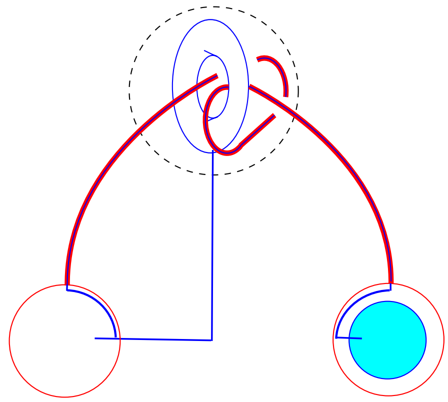

FIGURE 21. Teal edge isotoped into red tube

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MARTIN SCHARLEMANN, MATHEMATICS DEPARTMENT, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106-3080 USA

Email address: `mgscharl@math.ucsb.edu`