AN OVERVIEW OF PROPERTY 2R

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Abstract. The celebrated Property R Conjecture, affirmed by David Gabai [Ga], can be viewed as the first stage of a sequence of conjectures culminating in what has been called the Generalized Property R Conjecture. This conjecture is relevant to the study of outstanding problems in both 3-manifolds (specifically, links in S^3) and 4-manifolds (specifically, the Schoenflies Conjecture and the smooth Poincare Conjecture). Here we give an overview of part of forthcoming work of R. Gompf, A. Thompson and the author which considers the next stage in such a progression, called the Property 2R Conjecture.

It is shown that the lowest genus counterexample (if any exists) cannot be fibered. Exploiting Andrews-Curtis type considerations on presentations of the trivial group, it is argued that one of the simplest possible candidates for a counterexample, the square knot, probably is one. This suggests there is a genus one counterexample, though we have so far been unable to identify it. Finally, we note that the counterexample need not be an obstacle to the sort of 4-manifold consequences towards which the Generalized Property R Conjecture is aimed.

1. Generalizing Property R

A major development in knot theory during the 1980's was David Gabai's proof of the Property R theorem [Ga]:

Theorem 1.1 (Property R)**.** *If* 0*-framed surgery on a knot* $K \subset S^3$ *yields* $S^1 \times S^2$ *then* K *is the unknot.*

In the ensuing quarter century some effort has been made to sensibly generalize this conjecture, though very little public progress has been made. It is a particularly provocative conjecture because it is relevant to problems in both dimension 3 and dimension 4. Here we give a brief outline of some forthcoming results by R. Gompf, A. Thompson and the author on the question [GST], results that followed a 2007 meeting arranged by Mike Freedman at Microsoft's Station Q. Proofs can be found in [GST].

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There is a plausible way of trying to generalize Theorem 1.1 to links in S^3, but for more than one component so-called handle-slides are required. (The terminology is motivated by a related 4-dimensional picture.) Suppose U and V are two components of a framed link $L \subset S^3$. A handle-slide of U over V changes L to the link obtained by replacing U with a band sum $\overline{U}$ of U and a copy of V that has been pushed off of V by its framing.

Let $\#_n(S^1 \times S^2)$ denote the connected sum of n copies of $S^1 \times S^2$. The Generalized Property R conjecture (see [Ki, Problem 1.82]) says this:

Conjecture 1 (Generalized Property R)**.** *Suppose L is an integrally framed link of $n \geq 1$ components in S^3, and surgery on L via the specified framing yields $\#_n(S^1 \times S^2)$. Then there is a sequence of handle slides on L that converts L into a 0-framed unlink.*

Framing is not an issue: an elementary homology argument shows that any candidate must have framing 0 on all components (and also linking number 0 between any pair of components.) In the case $n = 1$ no slides are possible, so Conjecture 1 does indeed directly generalize Theorem 1.1. On the other hand, for $n > 1$ it is certainly necessary to include the possibility of handle slides. Figure 1 shows that 0-framed surgery on a certain link of the unknot with the square knot creates $\#_2(S^1 \times S^2)$. In a similar spirit, Figure 2 shows that even more complicated such framed links are easily created.

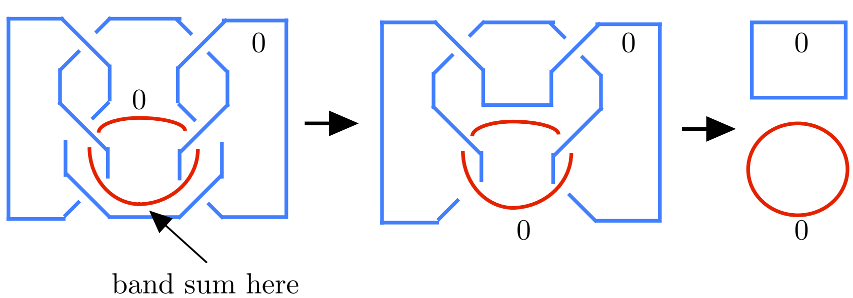

FIGURE 1

There is an immediate topological restriction on the link itself (see [Hi, Theorem 2]) a restriction that hints at the connection with 4-dimensional problems.

Proposition 1.2 (Hillman)**.** *Suppose L is a framed link of $n \geq 1$ components in S^3, and surgery on L via the specified framing yields $\#_n(S^1 \times S^2)$. Then L bounds a collection of n smooth 2- disks in a 4-dimensional homotopy ball bounded by S^3.*

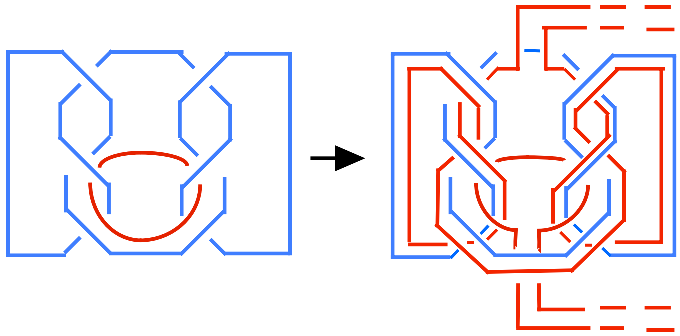

FIGURE 2

An equivalent way of stating the conclusion, following Freedman's proof of the 4-dimensional topological Poincare Conjecture [Fr], is that L (and so each component of L) is topologically slice in B^4.

The Generalized Property R Conjecture is a conjecture about framed links, but if we include in the conjecture the number of components and state it somewhat obliquely, it can be viewed as a sequence of conjectures about knots:

Definition 1.3. *A knot $K \subset S^3$ has* **Property nR** *if it does not appear among the components of any n-component counterexamples to the Generalized Property R conjecture.*

Conjecture 2 (Property nR Conjecture)**.** *All knots have Property nR.*

Thus the Generalized Property R conjecture for all n component links is equivalent to the Property nR Conjecture for all knots. Following Proposition 1.2 any non-slice knot has Property nR for all n. The main focus of our work has been on Property 2R.

2. Property 2R

To appreciate the role of handle-slides in the argument it is instructive to consider two very special cases of Property 2R. The first was shown to me by Alan Reid:

Proposition 2.1 (A. Reid)**.** *Suppose $L \subset S^3$ is a 2-component link with tunnel number 1. If surgery on L gives $\#_2(S^1 \times S^2)$ then L is the unlink of two components.*

Note that handle-slides (the new and necessary ingredient for Generalized Property R) do not arise. In contrast, Figure 1 shows that handle slides are needed in the proof of the following:

Proposition 2.2. *The unknot has Property 2R.*

That is, if surgery on a framed link of two components in which one component is the unknot gives $\#_2(S^1 \times S^2)$, then after handle-slides the link becomes the 0-framed unlink. The proof shows more: only handle-slides over the unknotted component are needed. That is, the unknotted component does not change during the sequence of handle-slides.

In contrast, the proof of the next result explicitly does require handle slides over *both* components of the link.

Theorem 2.3. *No smallest genus counterexample to Property 2R is fibered.*

In other words: Suppose surgery on a framed link of two components gives $\#_2(S^1 \times S^2)$, and one component of the link is a fibered knot U. Then, perhaps after handle-slides, at least one component of the link will have genus less than $genus(U)$. The proof makes use of the central result of [ST], which leads fairly directly to this preliminary observation that is interesting in its own right:

Lemma 2.4. *Suppose surgery on a framed link of two components $U, V \subset S^3$ gives $\#_2(S^1 \times S^2)$, and suppose U is a fibered knot. Then, perhaps after some slides over U, the component V lies on a fiber of U and the 0-framing of V in S^3 coincides with the framing given by the fiber.*

Following Lemma 2.4 it is natural to ask what properties V must have in the fiber in order that surgery on the pair U, V gives $\#_2(S^1 \times S^2)$. A surprising application of Heegaard splitting theory gives:

Proposition 2.5. *Suppose surgery on a framed link of two components $U, V \subset S^3$ gives $\#_2(S^1 \times S^2)$. Suppose further that*

- *U is a fibered knot*
- *V lies on a fiber F_- of U and*
- *the framing of V by the fiber is the 0-framing in S^3.*

Then, for $h : F_- \to F_-$ the fiber monodromy , $h(V)$ can be isotoped off of V in the closed *surface $F = F_- \cup_\partial D^2$.*

The distinction between the isotopy here taking place in the closed surface rather than the original punctured surface F_- could be crucial. For if it were not, the following proposition would guarantee that all genus two fibered knots have Property 2R, and this is regarded as highly unlikely for reasons which we will eventually discuss.

Proposition 2.6. *Suppose $U \subset S^3$ is a fibered knot, with fiber the punctured surface $F_- \subset S^3$ and monodromy $h_- : F_- \to F_-$. Suppose a* *knot $V \subset F_-$ has the property that 0-framed surgery on the link $U \cup V$ gives $\#_2(S^1 \times S^2)$ and $h_-(V)$ can be isotoped to be disjoint from V in F_-. Then either V is the unknot or genus$(F_-) \neq 1, 2$.*

In the special case of genus two fibered knots one can further show that, at the same time that $h(V)$ can be isotoped in the closed surface F to be disjoint from V, it will never be isotopic to V itself and, conversely, the properties we have shown suffice to characterize those curves V in the fiber which have the property that surgery on U, V yields $\#_2(S^1 \times S^2)$. That is:

Proposition 2.7. *Suppose $U \subset S^3$ is a genus two fibered knot and $V \subset S^3$ is a disjoint knot. Then 0-framed surgery on $U \cup V$ gives $\#_2(S^1 \times S^2)$ if and only if after possible handle-slides of V over U,*

(1) *V lies in a fiber of U;*
(2) *in the closed fiber F of the manifold M obtained by 0-framed surgery on U, $h(V)$ can be isotoped to be disjoint from V;*
(3) *$h(V)$ is not isotopic to V in F; and*
(4) *the framing of V given by F is the 0-framing of V in S^3.*

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We turn to the specific and very simple example of the genus two fibered knot called the square knot Q. It is the connected sum of the right-hand trefoil knot and the left-hand trefoil knot. There are many 2-component links containing Q so that surgery on the link gives $\#_2(S^1 \times S^2)$. Figure 1 shows (by sliding Q over the unknot) that the other component could be the unknot; Figure 2 shows (by instead sliding the unknot over Q) that the second component could be quite complicated. It turns out that, up to handle-slides of V over Q, there is an easy description of all two component links $Q \cup V$, so that surgery on $Q \cup V$ gives $\#_2(S^1 \times S^2)$. The critical ingredient in the characterization of V is the collection of properties listed in Proposition 2.7.

Let M be the 3-manifold obtained by 0-framed surgery on the square knot Q, so M fibers over the circle with fiber the closed genus 2 surface F. There is a simple picture of the monodromy $h : F \to F$ of the bundle M, obtained from a similar picture of the monodromy on the fiber of a trefoil knot, essentially by doubling it [Ro, Section 10.I]:

Regard F as obtained from two spheres by attaching 3 tubes between them. See Figure 3. There is an obvious period 3 symmetry $\rho : F \to F$ gotten by rotating $\frac{2\pi}{3}$ around an axis intersecting each sphere in two points, and a period 2 symmetry (the hyperelliptic involution) $\sigma : F \to F$ obtained by rotating around a circular axis that intersects each tube

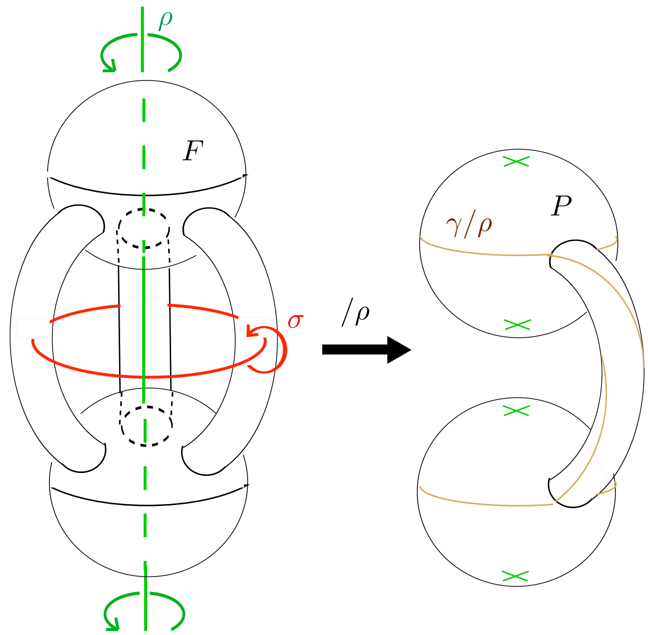

FIGURE 3

in two points. Then $h = \rho \circ \sigma = \sigma \circ \rho$ is an automorphism of F of order $2 \times 3 = 6$.

The quotient of F under the action of ρ is a sphere with 4 branch points, each of branching index 3. Let P be the 4-punctured sphere obtained by removing the branch points. A simple closed curve in P is essential if and only if it divides P into two twice-punctured disks. It is easy to see [GST] that there is a separating simple closed curve $\gamma \subset F$ that is invariant under σ and ρ, and hence under h, that separates F into two punctured tori F_R and F_L; the restriction of h to F_R or F_L is the monodromy of the trefoil knot. The quotient of γ under ρ is shown as the brown curve in Figure 3 .

Here then is the characterization:

Proposition 2.8. *Suppose $Q \subset S^3$ is the square knot with fiber $F_- \subset S^3$ and $V \subset S^3$ is a disjoint knot. Then 0-framed surgery on $Q \cup V$ gives $\#_2(S^1 \times S^2)$ if and only if, after perhaps some handle-slides of V over Q, V lies in F_- and ρ projects V homeomorphically to an essential simple closed curve in P.*

Essential simple closed curves $\bar{c}$ in P that are such homeomorphic projections are precisely those for which one branch point of F_L (or, equivalently, one branch point from F_R) lies on each side of $\bar{c}$. So another way of saying that ρ projects V homeomorphically to an essential simple closed curve in P is to say that V is the lift of an essential simple closed curve in P that separates one branch point of F_L (or, equivalently F_R) from the other.

Having established exactly what knots, combined with the square knot, can be surgered to get $\#_2(S^1 \times S^2)$, it would seem to be a straightforward task to show that these links do satisfy the Generalized Property R Conjecture. In fact the story now gets murky, as we try to integrate information from the theory of 4-manifolds.

3. The 4-manifold viewpoint: a non-standard handle structure on S^4

In [Go1], R. Gompf provided unexpected examples of handle structures on homotopy 4-spheres which do not obviously simplify to give the trivial handle structure on S^4. At least one family is highly relevant to the discussion above. This is example [Go1, Figure 1], reproduced here as the left side of Figure 4. (Setting $k = 1$ gives rise to the square knot.) A sequence of Kirby operations in [Go1, §2] shows that the resulting 4-manifold has boundary S^3.

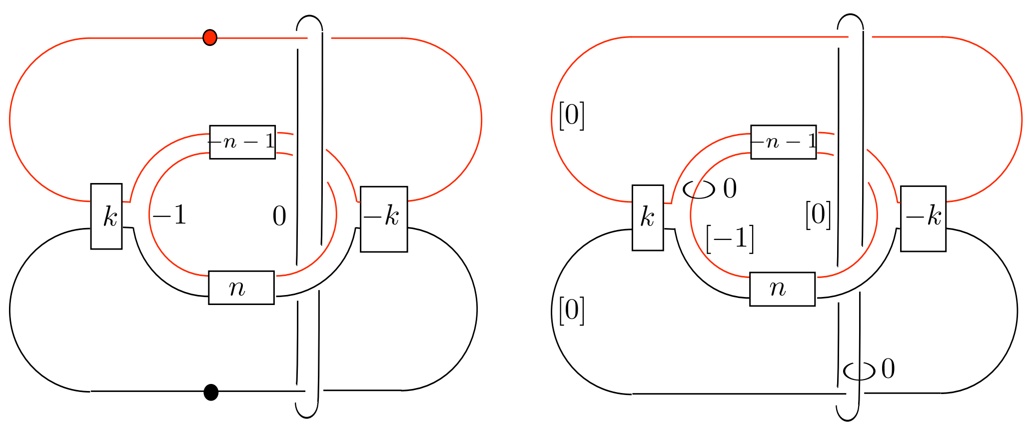

Figure 4

We will be interested in the 4-manifold that is the trace of the 2-handle surgeries, the manifold that lies between $\#_2(S^1 \times S^2)$ and S^3. If the 4-manifold is thought of as starting with S^3 to which two 2-handles are attached to get $\#_2(S^1 \times S^2)$ the construction is solidly in the context of this paper, for the picture becomes a link of two components, one of them the square knot.

Figures 5 (clockwise around the figure beginning at the upper left) and 6 show the end of the process; the middle 0-framed component becomes the square knot $Q \subset S^3$. (The other component becomes an interleaved connected sum of two torus knots, $V_n = T_{n,n+1} \# \overline{T_{n,n+1}}$.)

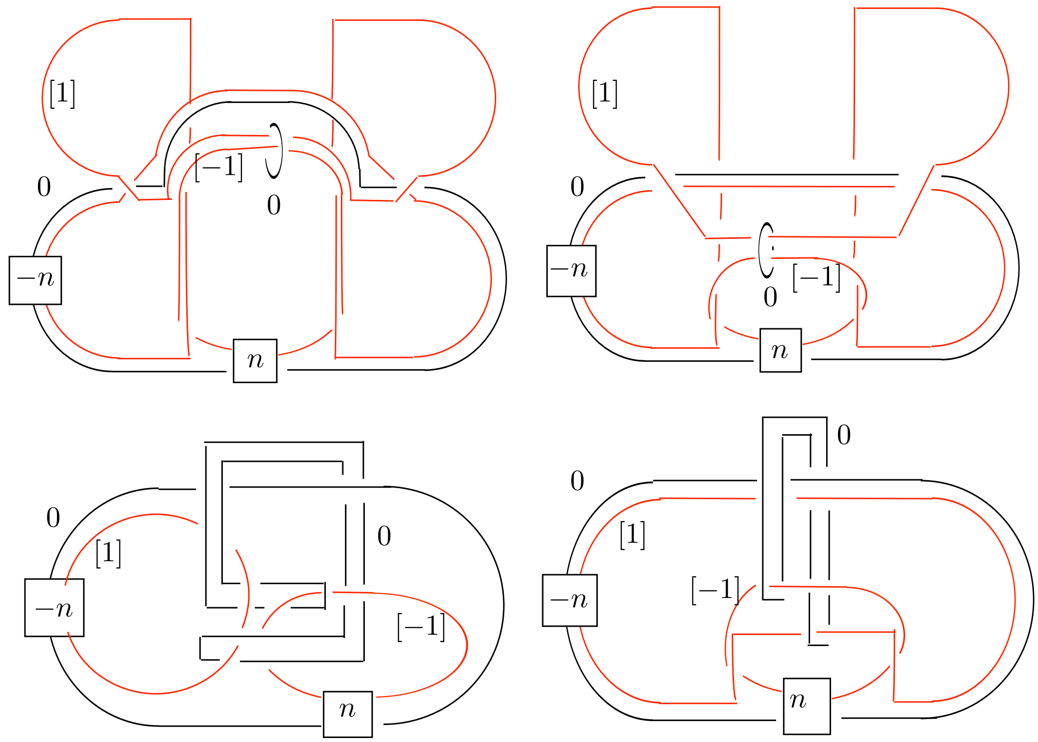

FIGURE 5

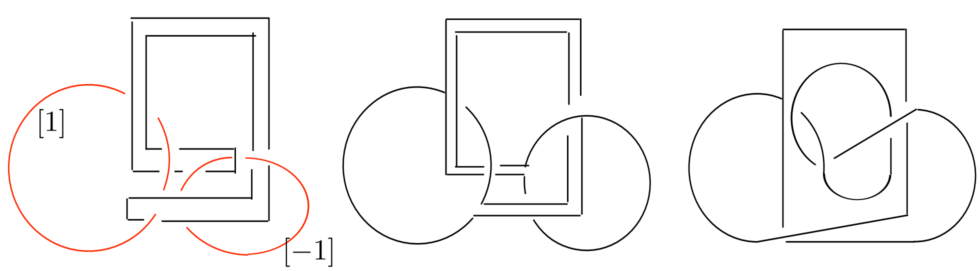

FIGURE 6

Two natural questions arise:

Question One: As described, V_n does not obviously lie on a Seifert surface for Q. According to Corollary 2.8, some handle slides of V_n over Q should alter V_n so that it is one of the easily enumerated curves that do lie on the Seifert surface, in particular it would be among those that are lifts of (half of) the essential simple closed curves in the 4-punctured sphere P. Which curves in P represent V_n for some n?

Question Two: Is each $Q \cup V_n, n \geq 3$, a counter-example to Generalized Property R?

This second question is motivated by Figure 4. As described in [Go1], the first diagram of that figure exhibits a simply connected 2-complex, presenting the trivial group as

$$\langle x, y \mid y = w^{-1}xw,\ x^{n+1} = y^n \rangle,$$

where w is some word in $x^{\pm 1}, y^{\pm 1}$ depending on k and equal to yx when $k = 1$. If the 2-component link of Figure 5 (after blowing down the two bracketed circles) can be changed to the unlink by handle slides, then the dual slides in Figure 4 will trivialize that picture, showing that the above presentation is Andrews-Curtis trivial. For $k = 1$, for example, this is regarded as very unlikely when $n \geq 3$. Since surgery on the link is $\#_2(S^1 \times S^2)$ by construction, this suggests an affirmative answer to Question Two, which (for any one n) would imply:

Conjecture 3. *The square knot does not have Property 2R.*

Although this news from the world of 4-manifolds is both puzzling and perhaps unwelcome, the 4-manifold perspective also suggests a weaker but more awkward version of Generalized Property R which would still provide the sort of 4-manifold results one would hope for:

Conjecture 4 (Weak Generalized Property R)**.** *Suppose L is a framed link of $n \geq 1$ components in S^3, and surgery on L yields $\#_n(S^1 \times S^2)$. Then, perhaps after adding a distant r-component 0-framed unlink and a set of s canceling Hopf pairs to L, there is a sequence of handle-slides that creates the distant union of an $n + r$ component 0-framed unlink with a set of s canceling Hopf pairs.*

Here a canceling Hopf pair is a Hopf link with one component of the link labeled with a dot and the other given framing 0. The dotted component represents a 1-handle and the 0-framed component represents the attaching circle for a canceling 2-handle. From the 4-manifold point of view adding a canceling Hopf pair makes no difference to the topology of the underlying 4-manifold since it denotes a pair of canceling 1- and 2- handles. But it can destroy the Andrews-Curtis obstruction, since adding a canceling Hopf pair introduces a new relator that is obviously trivial.

Definition 3.1. *A knot $K \subset S^3$ has* **Weak Property nR** *if it does not appear among the components of any n-component counterexample to the Weak Generalized Property R conjecture.*

The Weak Generalized Property R Conjecture is closely related to the Smooth (or PL) 4-Dimensional Poincaré Conjecture, that every homotopy 4-sphere is actually diffeomorphic to S^4. For a precise statement, we restrict attention to homotopy spheres that admit handle decompositions without 1-handles.

Proposition 3.2. *The Weak Generalized Property R Conjecture is equivalent to the Smooth 4-Dimensional Poincaré Conjecture for homotopy spheres that admit handle decompositions without 1-handles.*

While there are various known ways of constructing potential counterexamples to the Smooth 4-Dimensional Poincaré Conjecture, each method is known to produce standard 4-spheres in many special cases. (The most recent developments are [Ak], [Go2].) Akbulut's recent work [Ak] has eliminated the only promising potential counterexamples currently known to admit handle decompositions without 1-handles. For 3-dimensional renderings of the full Smooth 4-Dimensional Poincaré Conjecture and other related conjectures from 4-manifold theory, see [FGMW].

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