STRUCTURED STRONG LINEARIZATIONS FROM FIEDLER PENCILS WITH REPETITION I*

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Abstract. In many applications, the polynomial eigenvalue problem, $P(\lambda)x = 0$, arises with $P(\lambda)$ being a structured matrix polynomial (for example, palindromic, symmetric, skew-symmetric). In order to solve a structured polynomial eigenvalue problem it is convenient to use strong linearizations with the same structure as $P(\lambda)$ to ensure that the symmetries in the eigenvalues due to that structure are preserved in numerical computations. In this paper we characterize all the pencils in the family of the Fiedler pencils with repetition, introduced by Vologiannidis and Antoniou [25], associated with a square matrix polynomial $P(\lambda)$ that are block-symmetric for every matrix polynomial $P(\lambda)$. We show that this family of pencils is precisely the set of all Fiedler pencils with repetition that are symmetric when $P(\lambda)$ is. When some generic nonsingularity conditions are satisfied, these pencils are strong linearizations of $P(\lambda)$. In particular, our family strictly contains the standard basis for $\mathbb{DL}(P)$, a k-dimensional vector space of symmetric pencils introduced by Mackey, Mackey, Mehl, and Mehrmann [20].

Key words. Symmetric linearization, Fiedler pencils with repetition, matrix polynomial, companion form, polynomial eigenvalue problem.

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1. Introduction.

Let

$$P(\lambda) = A_k\lambda^k + A_{k-1}\lambda^{k-1} + \cdots + A_0 \tag{1.1}$$

be a matrix polynomial of degree $k \geq 2$, where the coefficients A_i are $n \times n$ matrices with entries in an arbitrary field $\mathbb{F}$.

A matrix pencil $L(\lambda) = \lambda L_1 - L_0$, with $L_1, L_0 \in M_{kn}(\mathbb{F})$, is a *linearization* of $P(\lambda)$ (see [12]) if there exist two unimodular matrix polynomials (i.e. matrix polynomials with constant nonzero determinant), $U(\lambda)$ and $V(\lambda)$, such that

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}.$$

Here and hereafter I_m denotes the $m \times m$ identity matrix. By I we denote the identity matrix whose size is clear from the context.

Linearizations of a matrix polynomial $P(\lambda)$ share the finite elementary divisors of $P(\lambda)$, among other important properties. Beside other applications, linearizations of matrix polynomials [12] are used in the study of the polynomial eigenvalue problem $P(\lambda)x = 0$. In the classical approach to this problem the original matrix polynomial $P(\lambda)$ is replaced by a matrix pencil $L_P(\lambda)$ of larger size with the same eigenvalues as $P(\lambda)$. Then, the standard methods for linear eigenvalue problems are applied.

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The *reversal* of the matrix polynomial $P(\lambda)$ in (1.1) is the matrix polynomial obtained by reversing the order of the coefficient matrices, that is,

$$rev(P(\lambda)) := \sum_{i=0}^{k} \lambda^i A_{k-i}.$$

A linearization $L(\lambda)$ of a matrix polynomial $P(\lambda)$ is called a *strong linearization* of $P(\lambda)$ if $rev(L(\lambda))$ is also a linearization of $rev(P(\lambda))$. Strong linearizations of $P(\lambda)$ have the same finite and infinite elementary divisors [11] as $P(\lambda)$. Moreover, any linearization with the same infinite elementary divisors as $P(\lambda)$ is a strong linearization [9, Theorem 4.1].

From the numerical point of view, it is not enough to have a strong linearization of a matrix polynomial. In any computational problem it is important to take into account its conditioning, i.e. its sensitivity to perturbations. In particular, when solving a polynomial eigenvalue problem, it is important to consider the eigenvalue condition number. More precisely, the relation of the condition number of an eigenvalue of the linearization and of the same eigenvalue of the matrix polynomial is relevant. It is known that different linearizations for a given polynomial eigenvalue problem can have very different conditioning [14, 23, 24]. This implies that numerical methods may produce quite different results for different linearizations. Therefore, it is convenient to have available a large class of structured linearizations that can be constructed easily and from which a linearization as well-conditioned as the original problem can be chosen.

For each matrix polynomial $P(\lambda)$, many different linearizations can be constructed but, in practice, those sharing the structure of $P(\lambda)$ are the most convenient from the theoretical and computational point of view, since the structure of $P(\lambda)$ often implies some symmetries in its spectrum, which are meaningful in physical applications and that can be destroyed by numerical methods when the structure is ignored [24]. For example, if a matrix polynomial is real symmetric or Hermitian, we have simultaneously that its spectrum is symmetric with respect to the real axis and the sets of left and right eigenvectors coincide. Thus, it is important to construct linearizations that reflect the structure of the original problem. In the literature [2, 3, 8, 15, 16, 21, 22] different kinds of structured linearizations have been considered: palindromic, symmetric, skew-symmetric, alternating, etc. Among the structured linearizations, those that are strong and can be easily constructed from the coefficients of the matrix polynomial $P(\lambda)$ are of particular interest [1, 2, 10, 25, 20], more precisely, those strong linearizations $L_P(\lambda) = \lambda L_1 - L_0$ such that each $n \times n$ block of L_1 and L_0 is either 0_n, $\pm I_n$, or $\pm A_i$, for $i = 0, 1, ..., k$, when L_1 and L_0 are viewed as $k \times k$ block matrices. There are some well-known families of linearizations with this property: Fiedler pencils ([1],[7]), generalized Fiedler pencils ([1],[5]), and Fiedler pencils with repetition (FPR) ([25]). We observe that, when at least one of the coefficients A_0 or A_k of the matrix polynomial $P(\lambda)$ of the form (1.1) is singular, not all Fiedler pencils with repetition $L(\lambda)$ associated with $P(\lambda)$ are strong linearizations of $P(\lambda)$. However, some conditions on $P(\lambda)$ and $L(\lambda)$ ensure that $L(\lambda)$ is a strong linearization of $P(\lambda)$, as mentioned later.

In a previous paper [3], we constructed a family of strong linearizations from the family of Fiedler pencils with repetition that preserve the palindromic (in case the matrix polynomial has odd degree) structure of the matrix polynomial $P(\lambda)$. Here we will consider the symmetric case. In the second part of this paper [4], we will study the skew-symmetric and T-alternating cases. The study of the symmetric case provides the tools to study the latter two cases.

In the literature, symmetric linearizations for symmetric matrix polynomials can be found [1, 2, 15, 17, 18, 19, 20, 25]. We are interested in symmetric linearizations which are easily constructed from the coefficients of the matrix polynomial. Notice that, by Lemma 5.2, for $k > 1$, there are no symmetric linearizations in the family of Fiedler pencils and the symmetric linearizations in the family of generalized Fiedler pencils only includes a few pencils [1, 2]. Some examples of symmetric strong linearizations in the family of Fiedler pencils with repetition associated with a symmetric matrix polynomial $P(\lambda)$ of degree k are given in [25], where this family is introduced. We note that in that paper it is shown that the family of FPR includes k symmetric linearizations already presented in [17, 18, 19], which form the standard basis of the k-dimensional vector space of symmetric pencils $\mathbb{DL}(P) = \mathbb{L}_1(P) \cap \mathbb{L}_2(P)$ studied in [20, 15]. If the matrix coefficients A_0 and A_k of $P(\lambda)$ are nonsingular, these pencils are strong linearizations of $P(\lambda)$. Note that, although any pencil in $\mathbb{DL}(P)$ is symmetric when $P(\lambda)$ is, it is not necessarily a strong linearization of $P(\lambda)$. In fact, it has been shown that when $P(\lambda)$ is regular, almost all pencils in $\mathbb{DL}(P)$ are strong linearizations of $P(\lambda)$ [20]. However, if $P(\lambda)$ is singular, no pencil in $\mathbb{DL}(P)$ is [6].

As shown in Example 8 in [25], the family of symmetric strong linearizations among the FPR includes linearizations that had not appeared in the literature before, in particular that are not in $\mathbb{DL}(P)$. While in [25] only a few examples were constructed, in this paper we characterize all the pencils in the family of Fiedler pencils with repetition which are symmetric when the associated matrix polynomial $P(\lambda)$ is. Although not every pencil in this family is a strong linearization of $P(\lambda)$, we give conditions on the pencils and $P(\lambda)$ under which they are. In particular, when A_0 and A_k are both nonsingular, the family of symmetric FPR linearizations that we construct are strong linearizations of $P(\lambda)$ and extends the standard basis of the space $\mathbb{DL}(P)$ significantly, as Example 5.6 shows for $k = 4$. Notice that in this case we get ten pencils that are distinct if $A_k \neq I$ and $A_0 \neq -I$. In the worst scenario, when $A_k = I$ and $A_0 = -I$, we still get six distinct pencils. We would also like to point out that, applying Theorem 5.3 for $k \geq 3$, we can produce examples of strong linearizations that are not in $\mathbb{L}_1(P)$ and, therefore, that are not in $\mathbb{DL}(P)$. This can be checked by using the shifted sum characterization of pencils in $\mathbb{L}_1$ given in [20]. It remains an open problem to determine the exact number of distinct symmetric FPR for each value of k, although it is clear that for $k \geq 4$ this number is always greater than the degree k of the matrix polynomial. We finally highlight that if k is odd, the family of symmetric FPR always contains strong linearizations of $P(\lambda)$, even for singular matrix polynomials. The conditioning of linearizations in $\mathbb{DL}(P)$ is, to some extent, well understood [13]. The fact that the pencils that we study extend the standard basis of $\mathbb{DL}(P)$ is also promising from this point of view.

The paper is organized as follows. In Section 2, we introduce some general definitions and results regarding index tuples. Some notation used throughout the paper is also presented. In Section 3 we focus on symmetric index tuples. More precisely, we characterize the symmetric tuples that will be relevant in the construction of the symmetric (Hermitian) linearizations in the family of the Fiedler pencils with repetition. In Section 4 we introduce this FPR family and give some related results that will be needed in Section 5, where we give a description of the Fiedler pencils with repetition that are symmetric (Hermitian) strong linearizations when the matrix polynomial $P(\lambda)$ is. Additionally we provide a characterization of the FPR that are block-symmetric for all FPR. This characterization is crucial for the construction of the skew-symmetric and T-alternating strong linearizations in the second part of this paper. We close this part of the paper with a summary of the main results obtained.

2. Index Tuples. We call an *index tuple* any ordered tuple whose entries are integers.

In this section we introduce some definitions and results for index tuples. In particular, we define an equivalence relation in the set of index tuples and give a canonical form under this equivalence relation. We also give some notation that will be used throughout the paper.

2.1. General definitions and notation. For the purposes of this paper, it is important to distinguish between index tuples in which the entries are repeated or not. This justifies the following definition.

DEFINITION 2.1. *Let* $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ *be an index tuple. We say that* $\mathbf{t}$ *is* simple *if* $i_j \neq i_l$ *for all* $j, l \in \{1, 2, \ldots, r\}$, $j \neq l$.

If i, j are integers such that $j \geq i$, we denote by $(i : j)$ the tuple $(i, i+1, i+2, \ldots, j)$. If $j < i$, $(i : j)$ denotes the empty tuple. We will refer to the simple index tuple $(i : j)$, $j \geq i$, consisting of consecutive integers, as a *string*.

If i, j are integers such that $j \leq i$, we denote by $(i :_\downarrow j)$ the tuple $(i, i-1, i-2, \ldots, j)$. If $j > i$, $(i :_\downarrow j)$ denotes the empty tuple.

DEFINITION 2.2. *Let* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *be two index tuples. We say that* $\mathbf{t}_1$ *is a* subtuple *of* $\mathbf{t}_2$ *if* $\mathbf{t}_1 = \mathbf{t}_2$ *or if* $\mathbf{t}_1$ *can be obtained from* $\mathbf{t}_2$ *by deleting some indices in* $\mathbf{t}_2$. *If* $i_1, \ldots, i_r$ *are distinct indices, we call the subtuple of* $\mathbf{t}_1$ *with indices from* $\{i_1, \ldots, i_r\}$ *the subtuple of* $\mathbf{t}_1$ *obtained by deleting from* $\mathbf{t}_1$ *all indices distinct from* $i_1, \ldots, i_r$.

EXAMPLE 2.3. *Let* $\mathbf{t} = (1, 2, 1, 3, 2, 3)$ *be viewed as a tuple with indices from* $\{1, 2, 3, 4\}$. *The subtuple of* $\mathbf{t}$ *with indices from* $\{1, 2\}$ *is* $(1, 2, 1, 2)$; *the subtuple of* $\mathbf{t}$ *with indices from* $\{3, 4\}$ *is* $(3, 3)$.

Note that in a subtuple of an index tuple, the indices keep their original relative positions, that is, the order of the indices in the subtuple is not altered with respect to the order of those indices in the original tuple.

If $\mathbf{t} = (i_1, \ldots, i_r)$ is an index tuple and a is an integer, we denote by $a + \mathbf{t}$ the index tuple $(a + i_1, \ldots, a + i_r)$.

Given a tuple $\mathbf{t}$, we call the number of elements in $\mathbf{t}$ the *length of* $\mathbf{t}$ and denote it by $|t|$. We denote by $\mathbf{t}[j]$ the tuple obtained from $\mathbf{t}$ by deleting the last j elements (counting from right to left).

DEFINITION 2.4. *Let* $\mathbf{t} = (a : b)$ *be a string and* $l = |\mathbf{t}|$. *If* $l > 1$, *we call the* reverse-complement *of* $\mathbf{t}$ *the index tuple* $\mathbf{t}_{rev_c} := (\mathbf{t}[1], \ldots, \mathbf{t}[l-1])$. *If* $l = 1$, *the reverse-complement of* $\mathbf{t}$ *is empty.*

EXAMPLE 2.5. *The reverse-complement of* $\mathbf{t} = (0 : 6)$ *is* $\mathbf{t}_{rev_c} = (0 : 5, 0 : 4, 0 : 3, 0 : 2, 0 : 1, 0)$; *the reverse-complement of* $\mathbf{t} = (0)$ *is empty.*

DEFINITION 2.6. *Given an index tuple* $\mathbf{t} = (i_1, \ldots, i_r)$, *we define the* reversal tuple *of* $\mathbf{t}$ *as* $\mathrm{rev}(\mathbf{t}) := (i_r, \ldots, i_1)$.

Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be two index tuples. Some immediate properties of the reversal operation are:

- $rev(rev(\mathbf{t}_1)) = \mathbf{t}_1$,
- $rev(\mathbf{t}_1, \mathbf{t}_2) = (rev(\mathbf{t}_2), rev(\mathbf{t}_1))$.

2.2. Equivalence of tuples. We define an equivalence relation in the set of index tuples with indices from a given set of either nonnegative or negative integers.

DEFINITION 2.7. *We say that two nonnegative indices* i, j commute *if* $|i - j| \neq 1$.

Definition 2.8. *Let* $\mathbf{t}$ *and* $\mathbf{t}'$ *be two index tuples of nonnegative integers. We say that* $\mathbf{t}$ *is obtained from* $\mathbf{t}'$ *by a* transposition *if* $\mathbf{t}$ *is obtained from* $\mathbf{t}'$ *by interchanging two commuting indices in adjacent positions.*

Definition 2.9. *Given two index tuples* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *of nonnegative integers, we say that* $\mathbf{t}_1$ *is* equivalent *to* $\mathbf{t}_2$ *if* $\mathbf{t}_2$ *can be obtained from* $\mathbf{t}_1$ *by a sequence of transpositions. If* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *are index tuples of negative indices, we say that* $\mathbf{t}_1$ *is equivalent to* $\mathbf{t}_2$ *if* $-\mathbf{t}_1$ *is equivalent to* $-\mathbf{t}_2$*. If* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *are equivalent index tuples, we write* $\mathbf{t}_1 \sim \mathbf{t}_2$*,*

Note that the relation $\sim$ is an equivalence relation.

Example 2.10. *The index tuples* $\mathbf{t}_1 = (2, 5, 3, 1, 4)$ *and* $\mathbf{t}_2 = (5, 2, 3, 4, 1)$ *are equivalent.*

Remark 2.11. *Note that if* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *are equivalent tuples with indices from* $\{i, i+1\}$*, where* i *is a nonnegative integer, then* $\mathbf{t}_1 = \mathbf{t}_2$*.*

Observe that, if j is a positive (resp. negative) integer and $\mathbf{t}_1$ and $\mathbf{t}_2$ are equivalent index tuples of nonnegative (resp. negative) indices, then so are $j + \mathbf{t}_1$ and $j + \mathbf{t}_2$.

The next proposition will be very useful in the proofs of our results.

Proposition 2.12. *Let* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *be two index tuples with indices from a set* S *of nonnegative (resp. negative) integers. Then,* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *are equivalent if and only if, for each* $i \in S$*, the subtuples of* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *with indices from* $\{i, i+1\}$ *are the same.*

Proof. If $\mathbf{t}_1$ and $\mathbf{t}_2$ are equivalent then they contain the same indices with the same multiplicities, and, since i and $i+1$ do not commute, the stated subtuples are the same. For the converse, assume that $\mathbf{t}_1$ and $\mathbf{t}_2$ are not equivalent. If $\mathbf{t}_1$ and $\mathbf{t}_2$ do not have the same indices, clearly for some $i \in S$ the subtuples with indices from $\{i, i+1\}$ are distinct. Now suppose that $\mathbf{t}_1$ and $\mathbf{t}_2$ have the same indices. Let k be the first position (starting from the left) in which $\mathbf{t}_1$ and $\mathbf{t}_2$ differ and no transposition applied to the indices of $\mathbf{t}_2$ to the right of position $k-1$ can transform the index in position k into the corresponding index in $\mathbf{t}_1$, say i. Since, by applying transpositions on $\mathbf{t}_2$, we cannot find an equivalent tuple with i in position k (and the elements in the positions before k are equal in both tuples) this means that $i-1$ or $i+1$ appears to the right of position $k-1$ and to the left of the first i after position k in $\mathbf{t}_2$. But this implies that either the subtuples of $\mathbf{t}_1$ and $\mathbf{t}_2$ with indices from $\{i, i-1\}$ or the subtuples of $\mathbf{t}_1$ and $\mathbf{t}_2$ with indices from $\{i, i+1\}$ are different. □

The next example illustrates the application of Proposition 2.12.

Example 2.13. *Consider the tuples* $\mathbf{t}_1 = (1, 5, 4, 2)$ *and* $\mathbf{t}_2 = (5, 1, 2, 4)$ *with indices from* $S = \{1, 2, 4, 5\}$*. For each* $i \in S$*, the subtuples of* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *with indices from* $\{i, i+1\}$ *coincide and are given by* $(1, 2)$ *if* $i = 1$*,* (2) *if* $i = 2$*,* $(5, 4)$ *if* $i = 4$*, and* (5) *if* $i = 5$*. Thus, by Proposition 2.12,* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *are equivalent. Now consider the tuples* $\mathbf{t}_1 = (5, 6, 25)$ *and* $\mathbf{t}_2 = (5, 6, 30)$ *with indices from* $S = \{5, 6, 25, 30\}$*. Clearly, the subtuples of* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *with indices from* $\{i, i+1\}$*, when* $i = 25$*, do not coincide. Thus, by Proposition 2.12,* $\mathbf{t}_1$ *and* $\mathbf{t}_2$ *are not equivalent.*

The next result is an easy consequence of the previous proposition and will be used in the proofs of our results.

Lemma 2.14. *Let* $\mathbf{q}$ *be a permutation of* $\{0, 1, \dots, h\}$*,* $h \geq 0$*, and* $\mathbf{l}_q, \mathbf{r}_q, \mathbf{l}'_q, \mathbf{r}'_q$ *be tuples with indices from* $\{0, 1, \dots, h-1\}$*. Then,* $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (\mathbf{l}'_q, \mathbf{q}, \mathbf{r}'_q)$ *if and only if* $\mathbf{l}_q \sim \mathbf{l}'_q$ *and* $\mathbf{r}_q \sim \mathbf{r}'_q$*.*

Proof. Clearly, if $\mathbf{l}_q \sim \mathbf{l}'_q$ and $\mathbf{r}_q \sim \mathbf{r}'_q$ then $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (\mathbf{l}'_q, \mathbf{q}, \mathbf{r}'_q)$. Now we prove the converse. Suppose that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (\mathbf{l}'_q, \mathbf{q}, \mathbf{r}'_q)$. We prove that $\mathbf{r}_q \sim \mathbf{r}'_q$. The proof of $\mathbf{l}_q \sim \mathbf{l}'_q$ is similar. By Proposition 2.12, it is enough to show that, for any index $i \in \{0, \dots, h-1\}$, the subtuples of $\mathbf{r}_q$ and $\mathbf{r}'_q$ with indices from $\{i, i+1\}$ are the same. First we prove that $\mathbf{r}_q$ and $\mathbf{r}'_q$ have precisely the same indices.

In order to get a contradiction, assume that $i \leq h-1$ is the largest index such that the subtuples of $\mathbf{r}_q$ and $\mathbf{r}'_q$ with indices from $\{i\}$ have different lengths. Let m denote the number of indices equal to $i+1$ in $\mathbf{r}_q$ and $\mathbf{r}'_q$ (which can be 0). By Proposition 2.12, the subtuples of $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}'_q, \mathbf{q}, \mathbf{r}'_q)$ with indices from $\{i, i+1\}$ are the same, which gives a contradiction as the number of i's occurring to the right of the $(m+1)$th index equal to $i+1$, counting from the right, is different in both tuples.

Thus, we conclude that $\mathbf{r}_q$ and $\mathbf{r}'_q$ have precisely the same indices. Since, by Proposition 2.12, for each $i < h$, the subtuples of $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}'_q, \mathbf{q}, \mathbf{r}'_q)$ with indices from $\{i, i+1\}$ are the same, also the corresponding subtuples of $\mathbf{r}_q$ and $\mathbf{r}'_q$ are the same. Again by Proposition 2.12, the claim follows. □

We now extend the definition of commuting indices to index tuples.

Definition 2.15. *Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be two index tuples of nonnegative (resp. negative) integers. We say that $\mathbf{t}_1$ and $\mathbf{t}_2$* commute *if every index in $\mathbf{t}_1$ commutes with every index in $\mathbf{t}_2$.*

Note that, if $\mathbf{t}_1$ and $\mathbf{t}_2$ commute, then, for every index i in $\mathbf{t}_1$, $i-1$ and $i+1$ are not in $\mathbf{t}_2$. In particular, if $\mathbf{t}_1$ and $\mathbf{t}_2$ commute then $(\mathbf{t}_1, \mathbf{t}_2) \sim (\mathbf{t}_2, \mathbf{t}_1)$. Also, if $\mathbf{t}'_1$ and $\mathbf{t}'_2$ are subtuples of the commuting tuples $\mathbf{t}_1$ and $\mathbf{t}_2$, then $\mathbf{t}'_1$ and $\mathbf{t}'_2$ commute.

2.3. Successor Infix Property and column standard form. In this paper we are interested in index tuples satisfying the property called SIP that we define below. In the case of tuples of nonnegative indices satisfying this property, we also give a canonical form under the equivalence relation defined in the previous section. Expressing the index tuples satisfying the SIP in this canonical form is an important tool for proving our main results.

Definition 2.16. *[25, Definition 7] Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple of nonnegative (resp. negative) indices. Then, $\mathbf{t}$ is said to satisfy the* Successor Infix Property (SIP) *if for every pair of indices $i_a, i_b \in \mathbf{t}$, with $1 \leq a < b \leq r$, satisfying $i_a = i_b$, there exists at least one index $i_c = i_a + 1$ with $a < c < b$.*

Remark 2.17. *We observe that the SIP is invariant under equivalence. Moreover, any subtuple consisting of adjacent indices from an index tuple satisfying the SIP also satisfies the SIP. Additionally, if a tuple $\mathbf{t}$ satisfies the SIP, so does $rev(\mathbf{t})$ and $a + \mathbf{t}$ for any integer a.*

We now give a canonical form under equivalence for a tuple of nonnegative integers satisfying the SIP.

Definition 2.18. *Let $\mathbf{t}$ be an index tuple with indices from $\{0, 1, \ldots, h\}$, $h \geq 0$. Then $\mathbf{t}$ is said to be in* column standard form *if $\mathbf{t}$ is of the form*

$$(a_s : b_s, a_{s-1} : b_{s-1}, \ldots, a_2 : b_2, a_1 : b_1), \tag{2.1}$$

with $h \geq b_s > b_{s-1} > \cdots > b_2 > b_1 \geq 0$ and $0 \leq a_j \leq b_j$, for all $j = 1, \ldots, s$. We call each subtuple of consecutive integers $(a_i : b_i)$ a string *in $\mathbf{t}$.*

The connection between the column standard form of an index tuple and the SIP is shown in the following result.

Lemma 2.19. *Let $\mathbf{t} = (i_1, \ldots, i_r)$ be an index tuple with indices from $\{0, 1, \ldots, h\}$, $h \geq 0$. Then $\mathbf{t}$ satisfies the SIP if and only if $\mathbf{t}$ is equivalent to a tuple in column standard form.*

Proof. The "only if" statement follows from the proof of Theorem 2 in [25], which is an immediate consequence of Lemma 8 in that paper.

Now we prove the "if" statement. Since, by Remark 2.17, the SIP is invariant under equivalence, it is enough to see that a tuple in column standard form, as in

(2.1), satisfies the SIP. Suppose that the index i occurs twice in $csf(\mathbf{t})$. Then, there exists $j, k \in \{1, 2, \ldots, s\}$, with $j < k$, such that $a_j \leq i \leq b_j$ and $a_k \leq i \leq b_k$. Since $b_j < b_k$, we deduce that $i \leq b_j < b_k$, which implies that $a_k \leq i < i+1 \leq b_k$ and, thus, $i+1$ occurs on the right of i in the string $(a_k : b_k)$ in $csf(\mathbf{t})$.□

Taking into account Proposition 2.12, it follows that two tuples in column standard form are equivalent if and only if they coincide. We then have the following definition.

Definition 2.20. *The unique index tuple in column standard form equivalent to an index tuple* $\mathbf{t}$ *satisfying the SIP is called the* column standard form of $\mathbf{t}$ *and is denoted by* $csf(\mathbf{t})$.

Note that, in particular, if $\mathbf{t}$ is simple, then $\mathbf{t}$ satisfies the SIP and, therefore, $\mathbf{t}$ is equivalent to a tuple in column standard form. In this case, if $\mathbf{t}$ is a permutation of $\{0, 1, \ldots, h\}$, the column standard form of $\mathbf{t}$ has the form

$$csf(\mathbf{t}) = (t_w + 1 : h, t_{w-1} + 1 : t_w, \ldots, t_2 + 1 : t_3, t_1 + 1 : t_2, 0 : t_1)$$

for some positive integers $0 \leq t_1 < t_2 < \ldots < t_w < h$.

3. Symmetric Index Tuples. In this section we consider index tuples that are symmetric.

Definition 3.1. *We say that an index tuple* $\mathbf{t}$ *of nonnegative (resp. negative) indices is* symmetric *if* $\mathbf{t} \sim rev(\mathbf{t})$.

Note that if $\mathbf{t}$ is a symmetric index tuple and a is an integer such that $a + \mathbf{t}$ (resp. $a - \mathbf{t}$) is a tuple with either all nonnegative or all negative indices, then $a + \mathbf{t}$ (resp. $a - \mathbf{t}$) is also symmetric. Also, observe that any tuple equivalent to a symmetric tuple is also symmetric.

We are interested in symmetric tuples of the form $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfying the SIP, where $\mathbf{q}$ is a permutation of $\{0, 1, \ldots, h\}$, $\mathbf{l}_q$ and $\mathbf{r}_q$ are tuples (possibly not simple) with indices from $\{0, 1, \ldots, h-1\}$, and $(\mathbf{l}_q, \mathbf{r}_q)$ is also symmetric. We characterize the symmetric tuples of this kind and give a new canonical form under equivalence for them. The canonical form we present will be used in the construction of the symmetric linearizations.

3.1. The S and the SS properties. Here we introduce some properties of symmetric tuples that will be very useful in proving our results. We focus on nonnegative tuples but all the results in this section can be extended to tuples of negative indices as well.

Definition 3.2. *Let* $\mathbf{t}$ *be a tuple with indices from* $\{0, 1, ..., h\}$, $h \geq 0$. *We say that* $\mathbf{t}$ *has the* S property *if, for every index* $i \in \mathbf{t}$ *with* $i < h$, *the subtuple of* $\mathbf{t}$ *with indices from* $\{i, i+1\}$ *is symmetric. In particular, if for every index* $i \in \mathbf{t}$ *with* $i < h$ *such that* $i + 1 \in \mathbf{t}$, *the subtuple of* $\mathbf{t}$ *with indices from* $\{i, i+1\}$ *is of the form* $(i, i+1, i, i+1, ..., i+1, i)$ *or* $(i+1, i, i+1, ..., i, i+1)$, *we say that* $\mathbf{t}$ *has the* SS property.

Lemma 3.3. *Let* $\mathbf{t}$ *be a tuple with indices from* $\{0, 1, \ldots, h\}$, $h \geq 0$. *Then,* $\mathbf{t}$ *is symmetric if and only if* $\mathbf{t}$ *has the S property.*

Proof. If $\mathbf{t}$ is symmetric, then it is clear that $\mathbf{t}$ has the S property. Now assume that $\mathbf{t}$ is not symmetric in order to see that $\mathbf{t}$ does not satisfy the S property. Since $\mathbf{t}$ and $rev(\mathbf{t})$ are not equivalent, by Proposition 2.12, there is $i \in \mathbf{t}$ such that the subtuples of $\mathbf{t}$ and $rev(\mathbf{t})$ with indices from $\{i, i+1\}$ are distinct. Thus, the subtuple of $\mathbf{t}$ with indices from $\{i, i+1\}$ is not symmetric, which implies the result. □

In order to characterize the index tuples $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ which are symmetric and such that $(\mathbf{l}_q, \mathbf{r}_q)$ is also symmetric, we start by considering the case when $\mathbf{l}_q$ and $\mathbf{r}_q$ are disjoint tuples (that is, have no common indices).

Lemma 3.4. *Let $\mathbf{q}$ be a permutation of $\{0, 1, \ldots, h\}$, $h \geq 0$, and let $\mathbf{l}_q, \mathbf{r}_q$ be disjoint tuples with indices from $\{0, 1, ..., h-1\}$ such that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ or $(\mathbf{l}_q, \mathbf{r}_q)$ is symmetric. Then, $\mathbf{l}_q$ and $\mathbf{r}_q$ commute.*

Proof. We observe that there is no index i such that either $i \in \mathbf{l}_q$ and $i+1 \in \mathbf{r}_q$ or $i \in \mathbf{r}_q$ and $i+1 \in \mathbf{l}_q$, as, otherwise, the subtuple of $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ (or $(\mathbf{l}_q, \mathbf{r}_q)$) with indices from $\{i, i+1\}$ would not be symmetric, (as its first and last elements would be different), a contradiction by Lemma 3.3. ◻

Next we characterize, in terms of the SS property, the index tuples $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfying the SIP, with $\mathbf{l}_q$ and $\mathbf{r}_q$ disjoint and such that both $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}_q, \mathbf{r}_q)$ are symmetric. Note that if $(\mathbf{l}_q, \mathbf{r}_q)$ is symmetric and $\mathbf{l}_q$ and $\mathbf{r}_q$ are disjoint, from Lemmas 3.3 and 3.4, $\mathbf{l}_q$ and $\mathbf{r}_q$ are symmetric as well.

Lemma 3.5. *Let $\mathbf{q}$ be a permutation of $\{0, 1, \ldots, h\}$, $h \geq 0$, and let $\mathbf{l}_q, \mathbf{r}_q$ be disjoint tuples with indices from $\{0, 1, ..., h-1\}$ such that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfies the SIP. Then, $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}_q, \mathbf{r}_q)$ are both symmetric if and only if $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ has the SS property.*

Proof. Assume that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ has the SS property, which implies that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ has the S property. Then, by Lemma 3.3 and taking into account that, for every $i \in \{0, 1, \ldots, h-1\}$, the subtuple of $\mathbf{q}$ with indices from $\{i, i+1\}$ is of the form $(i, i+1)$ or $(i+1, i)$, the result follows.

Assume now that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}_q, \mathbf{r}_q)$ are both symmetric. Let $i \in \{0, 1, \ldots, h-1\}$. By the SIP, the subtuple $\mathbf{j}$ of $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ with indices from $\{i, i+1\}$ cannot have two adjacent i's. We next show that $\mathbf{j}$ cannot have two adjacent elements equal to $i+1$ either. Assume it does. Since $\mathbf{q}$ only contains one index $i+1$ and $\mathbf{l}_q$ and $\mathbf{r}_q$ are disjoint, we have either $i+1 \in \mathbf{l}_q$ or $i+1 \in \mathbf{r}_q$. Suppose that $i+1 \in \mathbf{r}_q$ (which implies that $i+1 \notin \mathbf{l}_q$). The argument is analogous if $i+1 \in \mathbf{l}_q$. By Lemma 3.4, $i \notin \mathbf{l}_q$. Let p be the smallest positive integer such that the entries in positions p and $p+1$ in $\mathbf{j}$ are $i+1$. Note that $p \geq 2$, since $\mathbf{q}$ contains one i and one $i+1$. Also, the entry in position $p-1$ in the subtuple of $\mathbf{r}_q$ with indices from $\{i, i+1\}$ (which is the entry in position $p+1$ in the subtuple $\mathbf{j}$) is $i+1$. Because $(\mathbf{l}_q, \mathbf{r}_q)$ is symmetric and $i, i+1 \notin \mathbf{l}_q$, by Lemma 3.3, the subtuple of $\mathbf{r}_q$ with indices from $\{i, i+1\}$ is symmetric. Thus, the $(p-1)$th element counting from right to left in $\mathbf{r}_q$ (and, therefore, in $\mathbf{j}$) is $i+1$. Since, by Lemma 3.3, the subtuple $\mathbf{j}$ is also symmetric, we would get that the entry in position $p-1$ in $\mathbf{j}$ is $i+1$, a contradiction. Thus, we have shown that, in the subtuple $\mathbf{j}$, the indices i and $i+1$ alternate. Since, by Lemma 3.3, the subtuple $\mathbf{j}$ is symmetric, the first and last entry of $\mathbf{j}$ are equal and the result follows. ◻

3.2. Admissible Tuples. Here we introduce the concept of admissible tuple which will allow us to find a new canonical form under equivalence for symmetric tuples of the form $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$. This canonical form will be very useful in the construction of symmetric linearizations based on FPR.

Definition 3.6. *Let $\mathbf{q}$ be a permutation of $\{0, 1, \ldots, h\}$, $h \geq 0$. We say that $\mathbf{q}$ is an* admissible tuple *relative to $\{0, 1, \ldots, h\}$ if the sequence of the lengths of the strings in $csf(\mathbf{q})$ is of the form $(2, ..., 2, l+1)$, where $l \geq 0$. We call l the index of $\mathbf{q}$.*

From now on, in order to make our statements clearer, we will associate to an arbitrary permutation of $\{0, 1, \ldots, h\}$ the letter $\mathbf{q}$ and to an admissible tuple the letter $\mathbf{w}$.

Example 3.7. *Here we give some examples of admissible index tuples.*

- $\mathbf{w}_1 = (6:7, 4:5, 0:3)$ *is an admissible tuple with index* 3 *relative to* $\{0, \dots, 7\}$.
- $\mathbf{w}_2 = (5:6, 3:4, 1:2, 0)$ *is an admissible tuple with index* 0 *relative to* $\{0, \dots, 6\}$.

Note that if $\mathbf{w}$ is an admissible tuple with index l relative to $\{0, 1, ..., h\}$, then h and l have the same parity.

In the next definition we construct an index tuple that, when appended to an admissible tuple, produces a symmetric index tuple. We use the notation for the reverse-complement of a tuple introduced in Definition 2.4.

DEFINITION 3.8. *(Symmetric complement) Let* $\mathbf{w}$ *be an admissible tuple with index* l *relative to* $\{0, 1, \dots, h\}$, $h \geq 0$. *We call the* symmetric complement of $\mathbf{w}$ *the tuple* $\mathbf{r}_w$ *defined as follows:*
- $\mathbf{r}_w = (h-1, h-3, ..., l+3, l+1, (0:l)_{rev_c})$, *if* $l \geq 1$,
- $\mathbf{r}_w = (h-1, h-3, ..., 1)$, *if* $l = 0$.

EXAMPLE 3.9. *The symmetric complements of the tuples* $\mathbf{w}_1$ *and* $\mathbf{w}_2$ *given in Example 3.7 are*

$$\mathbf{r}_{w_1} = (6, 4, 0:2, 0:1, 0) \qquad \textit{and} \qquad \mathbf{r}_{w_2} = (5, 3, 1),$$

respectively.

We next show that, if $\mathbf{w}$ is an admissible index tuple and $\mathbf{r}_w$ is the symmetric complement of $\mathbf{w}$, then $(\mathbf{w}, \mathbf{r}_w)$ is symmetric. We need the following auxiliary result.

PROPOSITION 3.10. *The reverse-complement of the string* $\mathbf{t} = (0:l)$, $l \geq 1$, *is symmetric and satisfies the SIP.*

Proof. Since $\mathbf{t}_{rev_c}$ is in column standard form, by Lemma 2.19, it satisfies the SIP. The rest of the proof is by induction on l. If $l = 1$, the result holds trivially. Now suppose that $l > 1$. Let $\mathbf{r}_i = (0:i)$, $i = 0, \dots, l-1$, so that $\mathbf{t}_{rev_c} = (\mathbf{r}_{l-1}, \dots, \mathbf{r}_0)$. Note that $(0:l-1)_{rev_c} = (\mathbf{r}_{l-2}, \dots, \mathbf{r}_0)$. By the induction hypothesis,

$$(\mathbf{r}_{l-2}, \dots, \mathbf{r}_0) \sim rev(\mathbf{r}_{l-2}, \dots, \mathbf{r}_0).$$

Then,

$$\begin{aligned} rev(\mathbf{t}_{rev_c}) &= (rev(\mathbf{r}_{l-2}, \dots, \mathbf{r}_0), rev(\mathbf{r}_{l-1})) \\ &\sim (\mathbf{r}_{l-2}, \dots, \mathbf{r}_0, l-1 :_{\downarrow} 0) \\ &\sim (\mathbf{r}_{l-2}, l-1, \mathbf{r}_{l-3}, l-2, \dots, \mathbf{r}_0, 1, 0) = \mathbf{t}_{rev_c}, \end{aligned}$$

where the last equivalence follows from the commutativity relations for indices. ☐

LEMMA 3.11. *Let* $\mathbf{w}$ *be an admissible tuple with index* l *relative to* $\{0, 1, \dots, h\}$, $h \geq 0$. *Let* $\mathbf{r}_w$ *be the symmetric complement of* $\mathbf{w}$. *Then,* $(\mathbf{w}, \mathbf{r}_w)$ *is symmetric and satisfies the SIP. Moreover,* $\mathbf{r}_w$ *is symmetric.*

Proof. The fact that $(\mathbf{w}, \mathbf{r}_w)$ satisfies the SIP follows from the definition of $\mathbf{r}_w$ and Proposition 3.10. Also, by Proposition 3.10 and taking into account the commutativity relations for indices, it follows that the tuple $\mathbf{r}_w$ is symmetric.

Now we show that $(\mathbf{w}, \mathbf{r}_w)$ is symmetric. Assume that $csf(\mathbf{w}) = (B_s, \dots, B_0)$, where B_i, $i = 0, \dots, s$, are the strings of $csf(\mathbf{w})$. We prove the result by induction on s. If $s = 0$ the claim follows from Proposition 3.10 taking into account that $(\mathbf{w}, \mathbf{r}_w)$ is the reverse complement of $(0:h+1)$. Now suppose that $s > 0$. Then, $\mathbf{w}' = (B_{s-1}, \dots, B_0)$ is an admissible tuple. Let $\mathbf{r}_{w'}$ be the symmetric complement of $\mathbf{w}'$. Note that $B_s = (h-1:h)$ and $\mathbf{r}_w \sim (\mathbf{r}_{w'}, h-1)$. Thus,

$$(\mathbf{w}, \mathbf{r}_w) \sim (h-1, h, \mathbf{w}', \mathbf{r}_{w'}, h-1)$$

So, we have

$$\begin{aligned}
rev(\mathbf{w}, \mathbf{r}_w) &\sim (h-1, rev(\mathbf{w}', \mathbf{r}_{w'}), h, h-1) \\
&\sim (h-1, \mathbf{w}', \mathbf{r}_{w'}, h, h-1) \\
&\sim (h-1, h, \mathbf{w}', \mathbf{r}_{w'}, h-1) \\
&\sim (\mathbf{w}, \mathbf{r}_w),
\end{aligned}$$

where the second equivalence follows from the induction hypothesis and the third equivalence follows because the largest index in $(\mathbf{w}', \mathbf{r}_{w'})$ is $h-2$ and, therefore, h commutes with any index in $(\mathbf{w}', \mathbf{r}_{w'})$. □

REMARK 3.12. *Note that, if $\mathbf{w}$ is an admissible tuple with indices from $\{0, 1, ..., h\}$, $h < k$, and $\mathbf{r}_w$ is the corresponding symmetric complement, then $(-k+\mathbf{w}, -k+\mathbf{r}_w)$ and $-k+\mathbf{r}_w$ are symmetric.*

3.3. Reduction to the Admissible Case. In this section we first prove that every symmetric index tuple of the form $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfying the SIP and such that $(\mathbf{l}_q, \mathbf{r}_q)$ is symmetric is equivalent to an index tuple of the form $(rev(\mathbf{t}), \mathbf{l}_q^*, \mathbf{q}, \mathbf{r}_q^*, \mathbf{t})$ with $\mathbf{l}_q^*$ and $\mathbf{r}_q^*$ disjoint. Then we show that $(\mathbf{l}_q^*, \mathbf{q}, \mathbf{r}_q^*)$ is equivalent to an index tuple of the form $(rev(\mathbf{t}'), \mathbf{w}, \mathbf{r}_w, \mathbf{t}')$, where $\mathbf{w}$ is an admissible tuple and $\mathbf{r}_w$ is the associated symmetric complement.

LEMMA 3.13. *Let $\mathbf{q}$ be a permutation of $\{0, 1, \ldots, h\}$, $h \geq 0$, and $\mathbf{l}_q, \mathbf{r}_q$ be tuples with indices from $\{0, 1, ..., h-1\}$ such that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfies the SIP. Suppose that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}_q, \mathbf{r}_q)$ are symmetric. Then, there exist unique (up to equivalence) index tuples $\mathbf{t}$, $\mathbf{l}_q^*$, $\mathbf{r}_q^*$, with indices from $\{0, \ldots, h-1\}$, such that $\mathbf{l}_q^*$ and $\mathbf{r}_q^*$ are disjoint and*

$$(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (rev(\mathbf{t}), \mathbf{l}_q^*, \mathbf{q}, \mathbf{r}_q^*, \mathbf{t}). \tag{3.1}$$

Moreover,

$$\mathbf{l}_q \sim (rev(\mathbf{t}), \mathbf{l}_q^*) \qquad \mathbf{r}_q \sim (\mathbf{r}_q^*, \mathbf{t}), \tag{3.2}$$

and $(\mathbf{l}_q^*, \mathbf{q}, \mathbf{r}_q^*)$ and $(\mathbf{l}_q^*, \mathbf{r}_q^*)$ are symmetric.

Proof. Assume that $\mathbf{l}_q$ and $\mathbf{r}_q$ are not disjoint, otherwise the existence claim follows with $\mathbf{t} = \emptyset$, $\mathbf{l}_q^* = \mathbf{l}_q$, and $\mathbf{r}_q^* = \mathbf{r}_q$. Let $\mathbf{l}_q = (i_1, \mathbf{l}_q')$ for some index i_1 and some index tuple $\mathbf{l}_q'$. Then, because $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ is symmetric, we have $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (i_1, \mathbf{l}_q', \mathbf{j}, i_1)$, for some tuple $\mathbf{j}$. Therefore, if $i_1 \notin \mathbf{r}_q$, then $\mathbf{j} \sim (\mathbf{q}', \mathbf{r}_q)$, where $\mathbf{q}'$ is the subtuple obtained from $\mathbf{q}$ by deleting the index i_1, and i_1 commutes with $\mathbf{r}_q$. Repeating this argument, we get that any index in $\mathbf{l}_q$ on the left of the first index in both $\mathbf{l}_q$ and $\mathbf{r}_q$, say j, should commute with $\mathbf{j}$. Thus, since $\mathbf{l}_q$ and $\mathbf{r}_q$ are not disjoint, we can commute the indices in $\mathbf{l}_q$ in order to have in the first position on the left an index in both $\mathbf{l}_q$ and $\mathbf{r}_q$. So, assume that $i_1 \in \mathbf{r}_q$. Moreover, because $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ is symmetric, we have $\mathbf{r}_q \sim (\mathbf{r}_q', i_1)$ for some index tuple $\mathbf{r}_q'$. Thus,

$$(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (i_1, \mathbf{l}_q', \mathbf{q}, \mathbf{r}_q', i_1).$$

Clearly, $(\mathbf{l}_q', \mathbf{q}, \mathbf{r}_q')$ and $(\mathbf{l}_q', \mathbf{r}_q')$ are symmetric. Applying this argument inductively, we get a tuple of the claimed form. By Lemma 2.14, (3.2) follows. By (3.1), (3.2) and Lemma 3.3, $(\mathbf{l}_q^*, \mathbf{q}, \mathbf{r}_q^*)$ and $(\mathbf{l}_q^*, \mathbf{r}_q^*)$ are symmetric.

Finally, we prove the uniqueness of $\mathbf{t}$, $\mathbf{l}_q^*$, $\mathbf{r}_q^*$. Suppose that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ is equivalent to another tuple $(rev(\mathbf{t}''), \mathbf{l}_q'', \mathbf{q}, \mathbf{r}_q'', \mathbf{t}'')$, where $\mathbf{l}_q''$ and $\mathbf{r}_q''$ are disjoint. By Lemma 2.14,

$\mathbf{r}_q \sim (\mathbf{r}''_q, \mathbf{t}'') \sim (\mathbf{r}^*_q, \mathbf{t})$. Analogously, $\mathbf{l}_q \sim (rev(\mathbf{t}''), \mathbf{l}''_q) \sim (rev(\mathbf{t}), \mathbf{l}^*_q)$. Since $\mathbf{l}^*_q$ and $\mathbf{r}^*_q$ (resp. $\mathbf{l}''_q$ and $\mathbf{r}''_q$) are disjoint, it follows that the indices in $\mathbf{t}$ (resp. $\mathbf{t}''$) are precisely those indices, counting multiplicities, that occur in both $\mathbf{l}_q$ and $\mathbf{r}_q$. Thus, $\mathbf{t}''$ and $\mathbf{t}$ have the same indices. Because $(\mathbf{r}''_q, \mathbf{t}'') \sim (\mathbf{r}^*_q, \mathbf{t})$, by Proposition 2.12, $\mathbf{t}'' \sim \mathbf{t}$ and $\mathbf{r}''_q \sim \mathbf{r}^*_q$. Similarly, it can be deduced that $\mathbf{l}''_q \sim \mathbf{l}^*_q$. □

EXAMPLE 3.14. *Let* $\mathbf{q} = (6, 3:5, 2, 0:1)$, $\mathbf{l}_q = (3:5, 1:2, 0:1)$ *and* $\mathbf{r}_q = (3:4, 2:3, 0:1)$. *It is easy to check that* $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ *and* $(\mathbf{l}_q, \mathbf{r}_q)$ *are both symmetric index tuples. Note that* $\mathbf{l}_q$ *and* $\mathbf{r}_q$ *are not disjoint. We have*

$$\mathbf{l}_q \sim ((3), (4:5, 1:2, 0:1)) \quad \textit{and} \quad \mathbf{r}_q \sim ((3:4, 2, 0:1), (3)).$$

Then,

$$(4:5, 1:2, 0:1) \sim ((4), (5, 1:2, 0:1)) \quad \textit{and} \quad (3:4, 2, 0:1) \sim ((3, 2, 0:1), (4)).$$

Also,

$$(5, 1:2, 0:1) \sim ((1), (5, 2, 0:1)) \quad \textit{and} \quad (3, 2, 0:1) \sim ((3, 2, 0), (1)).$$

After two more steps, we conclude that

$$\mathbf{l}_q \sim ((3, 4, 1, 2, 0), (5, 1)) \quad \textit{and} \quad \mathbf{r}_q \sim ((3), (0, 2, 1, 4, 3)).$$

Thus, (3.1) holds with $\mathbf{t} = (0, 2, 1, 4, 3)$, $\mathbf{l}^*_q = (5, 1)$, *and* $\mathbf{r}^*_q = (3)$.

In the previous lemma we expressed the tuple $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ in the form $(rev(\mathbf{t}), \mathbf{l}^*_q, q, \mathbf{r}^*_q, \mathbf{t})$ with $\mathbf{l}^*_q$ and $\mathbf{r}^*_q$ disjoint. Next we find an admissible tuple $\mathbf{w}$ such that $(\mathbf{l}^*_q, q, \mathbf{r}^*_q) \sim (rev(\mathbf{t}'), \mathbf{w}, \mathbf{r}_w, \mathbf{t}')$, where $\mathbf{r}_w$ is the symmetric complement of $\mathbf{w}$.

LEMMA 3.15. *Let* $\mathbf{q}$ *be a permutation of* $\{0, 1, \ldots, h\}$, $h \geq 0$, *and* $\mathbf{l}_q, \mathbf{r}_q$ *be disjoint tuples with indices from* $\{0, ..., h-1\}$. *Suppose that* $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ *is a symmetric tuple satisfying the SIP and* $(\mathbf{l}_q, \mathbf{r}_q)$ *is symmetric. Then, there exist an admissible tuple* $\mathbf{w}$ *relative to* $\{0, 1, ..., h\}$ *and an index tuple* $\mathbf{t}$ *with indices from* $\{0, \ldots, h-1\}$ *such that*

$$(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim (rev(\mathbf{t}), \mathbf{w}, \mathbf{r}_w, \mathbf{t}) \tag{3.3}$$

and

$$(\mathbf{l}_q, \mathbf{r}_q) \sim (rev(\mathbf{t}), \mathbf{r}_w, \mathbf{t}), \tag{3.4}$$

where $\mathbf{r}_w$ *is the symmetric complement of* $\mathbf{w}$.

Proof. In order to make the proof clearer, we assume $h \geq 2$. For $h < 2$ the result can be easily checked. The proof is by induction on the number of strings in $csf(\mathbf{q})$. Let $csf(\mathbf{q}) = (B_s, ..., B_1, B_0)$, where B_i, $i = 0, 1, .., s$, are the strings of $csf(\mathbf{q})$. Assume that $s = 0$, that is, $csf(\mathbf{q})$ has only one string. Then, $\mathbf{q} = (0:h)$, which is an admissible tuple. Note that, because of the SIP, $\mathbf{l}_q = \emptyset$. Let $\mathbf{r}'_q$ be the symmetric complement of $\mathbf{q}$. By Lemma 3.11, $(\mathbf{q}, \mathbf{r}'_q)$ satisfies the SIP, is symmetric, and $\mathbf{r}'_q$ is symmetric. We now show that $\mathbf{r}_q \sim \mathbf{r}'_q$, which implies the result. By Lemma 3.5, $(\mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{q}, \mathbf{r}'_q)$ satisfy the SS property. By Proposition 2.12, it is enough to show that for any $0 \leq i < h$, the subtuples of $\mathbf{r}_q$ and $\mathbf{r}'_q$ with indices from $\{i, i+1\}$ are the same. Note that in both tuples the first and last indices are equal to i. Because of the SIP, $h-1$ occurs exactly once in $\mathbf{r}_q$ and $\mathbf{r}'_q$. Then, $h-2$ occurs exactly twice. In general, $h-k$ occurs exactly k times in $\mathbf{r}_q$ and $\mathbf{r}'_q$. Thus, the claimed subtuples of $\mathbf{r}_q$ and $\mathbf{r}'_q$ with indices from $\{i, i+1\}$ coincide for each i, which implies, by Proposition 2.12, that $\mathbf{r}_q \sim \mathbf{r}'_q$.

Assume now that $s > 0$, that is, $csf(\mathbf{q})$ has more than one string. Note that, by Lemma 3.4, $\mathbf{l}_q$ and $\mathbf{r}_q$ commute. In the rest of the proof we use some notation introduced in Subsection 2.1.

Case 1: Suppose that $\mathbf{q} = (h :_\downarrow 0)$. Then, by Lemma 3.5, the subtuple of $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ with indices from $\{h-1, h\}$ must be of the form $(h-1, h, h-1)$, since $(h, h-1)$ is a subtuple of $\mathbf{q}$. Thus, $h-1 \in \mathbf{l}_q$. Note that, because of the SIP, $\mathbf{l}_q$ has at most one index equal to $h-1$. Applying Lemma 3.5 to the subtuple of $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ with indices from $\{h-2, h-1\}$, we deduce that the subtuple of $\mathbf{l}_q$ with indices from $\{h-2, h-1\}$ is $(h-2, h-1, h-2)$. By repeating this argument we conclude that $\mathbf{l}_q \sim (\mathbf{l}'_q, h-1 :_\downarrow 0)$, for some tuple $\mathbf{l}'_q \subset \{0, \ldots, h-2\}$. Since $\mathbf{r}_q$ and $\mathbf{l}_q$ are disjoint and have indices from $\{0, \ldots, h-1\}$, we deduce that $\mathbf{r}_q = \emptyset$. Because $\mathbf{l}_q = (\mathbf{l}_q, \mathbf{r}_q)$ is symmetric, it follows that

$$\mathbf{l}_q \sim (0 : h-2, \mathbf{l}''_q, h-1 :_\downarrow 0),$$

for some symmetric tuple $\mathbf{l}''_q \subset \{0, \ldots, h-3\}$. Note that, because of the SIP, $h-1, h-2 \notin \mathbf{l}''_q$. Therefore,

$$\begin{aligned}(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) &\sim (0 : h-2, \mathbf{l}''_q, h-1 :_\downarrow 0,\ h :_\downarrow 0) \\ &\sim (0 : h, \left(\mathbf{l}''_q, h-2 :_\downarrow 0\right), h-1 :_\downarrow 0).\end{aligned}$$

Because $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ is symmetric, so is $(\mathbf{l}''_q, h-2 :_\downarrow 0)$. Thus, the tuple $(\mathbf{l}''_q, h-2 :_\downarrow 0)$ satisfies the conditions of the theorem. By the induction hypothesis, there exist an admissible tuple $\mathbf{w}^*$ relative to $\{0, 1, ..., h-2\}$ and an index tuple $\mathbf{t}^*$ with indices from $\{0, 1, ..., h-3\}$ such that $(\mathbf{l}''_q, h-2 :_\downarrow 0) \sim (rev(\mathbf{t}^*), \mathbf{w}^*, \mathbf{r}^*_w, \mathbf{t}^*)$, where $\mathbf{r}^*_w$ is the symmetric complement of $\mathbf{w}^*$ and $\mathbf{l}''_q \sim (rev(\mathbf{t}^*), \mathbf{r}^*_w, \mathbf{t}^*)$. Then,

$$\begin{aligned}(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) &\sim (0 : h, rev(\mathbf{t}^*), \mathbf{w}^*, \mathbf{r}^*_w, \mathbf{t}^*, h-1 :_\downarrow 0) \\ &\sim (0 : h-2, rev(\mathbf{t}^*), (h-1 : h, \mathbf{w}^*), (\mathbf{r}^*_w, h-1), \mathbf{t}^*, h-2 :_\downarrow 0),\end{aligned}$$

and (3.3) holds with $\mathbf{t} = (\mathbf{t}^*, h-2 :_\downarrow 0)$, $\mathbf{r}_w = (h-1, \mathbf{r}^*_w)$ and $\mathbf{w} = (h-1 : h, \mathbf{w}^*)$. Condition (3.4) can be easily verified.

Case 2: Suppose that $B_s = (h)$ and $|B_i| > 1$ for some $i = 0, \ldots, s-1$. Let $j < s$ be the largest integer such that $|B_s| = \cdots = |B_{s-j}| = 1$. Then,

$$csf(\mathbf{q}) = (h :_\downarrow h-j,\ h-r : h-j-1, B_{s-j-2}, ..., B_0),$$

for some $r > j+1$. By Lemma 3.5, using an argument similar to that in Case 1, $\mathbf{l}_q \sim (\mathbf{l}'_q, h-1 :_\downarrow h-j-1)$, for some tuple $\mathbf{l}'_q \subset \{0, \ldots, h-2\}$. Note that, because of the SIP, $h-1 \notin \mathbf{l}'_q$. Since $(\mathbf{l}_q, \mathbf{r}_q)$ is symmetric and, by Lemma 3.4, $\mathbf{l}_q$ and $\mathbf{r}_q$ commute, we have that $\mathbf{l}_q$ is also symmetric, which implies

$$\mathbf{l}_q \sim (h-j-1 : h-2, \mathbf{l}''_q, h-1 :_\downarrow h-j-1),$$

for some symmetric tuple $\mathbf{l}''_q \subset \{0, \ldots, h-3\}$. Note that, by the SIP, $h-1 \notin \mathbf{l}''_q$. Also, for $j > 0$, again by the SIP, $h-2 \notin \mathbf{l}''_q$; when $j = 0$ the same conclusion follows from the symmetry of $\mathbf{l}_q$. Therefore,

$$\begin{aligned}(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) &\sim (h-j-1 : h-2, \mathbf{l}''_q, h-1 :_\downarrow h-j-1, \mathbf{q}, \mathbf{r}_q) \\ &\sim (h-j-1 : h, \left(\mathbf{l}''_q, h-2 :_\downarrow h-j-1, B'_{s-j-1}, B_{s-j-2}, ..., B_0, \mathbf{r}_q\right), h-1 :_\downarrow h-j-1),\end{aligned}$$

where $B'_{s-j-1} := B_{s-j-1}[1]$. Observe that, since $\mathbf{l}_q$ and $\mathbf{r}_q$ commute, so do $(h-1 :_\downarrow h-j-1)$ and $\mathbf{r}_q$. As $\mathbf{l}_q$ and $\mathbf{r}_q$ are disjoint and $h-1 \in \mathbf{l}_q$, by Lemma 3.4, $h-2, h-1 \notin \mathbf{r}_q$. Thus, the tuple $\left(\mathbf{l}''_q, h-2 :_\downarrow h-j-1, B'_{s-j-1}, B_{s-j-2}, ..., B_0, \mathbf{r}_q\right)$ satisfies the conditions of the theorem. By the induction hypothesis, there exist an admissible tuple $\mathbf{w}^*$ relative to $\{0, 1, ..., h-2\}$ and an index tuple $\mathbf{t}^*$ with indices from $\{0, 1, ..., h-3\}$ such that

$$\left(\mathbf{l}''_q, h-2 :_\downarrow h-j-1, B'_{s-j-1}, B_{s-j-2}, ..., B_0, \mathbf{r}_q\right) \sim (rev(\mathbf{t}^*), \mathbf{w}^*, \mathbf{r}^*_w, \mathbf{t}^*)$$

and $(\mathbf{l}''_q, \mathbf{r}_q) \sim (rev(\mathbf{t}^*), \mathbf{r}^*_w, \mathbf{t}^*)$, where $\mathbf{r}^*_w$ is the symmetric complement of $\mathbf{w}^*$. Then,

$$\begin{aligned}(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) &\sim (h-j-1 : h, rev(\mathbf{t}^*), \mathbf{w}^*, \mathbf{r}^*_w, \mathbf{t}^*, h-1 :_\downarrow h-j-1) \\ &\sim (h-j-1 : h-2, rev(\mathbf{t}^*), (h-1 : h, \mathbf{w}^*), (\mathbf{r}^*_w, h-1), \mathbf{t}^*, h-2 :_\downarrow h-j-1),\end{aligned}$$

and (3.3) holds with $\mathbf{t} = (\mathbf{t}^*, h-2 :_\downarrow h-j-1)$, $\mathbf{r}_w = (h-1, \mathbf{r}^*_w)$ and $\mathbf{w} = (h-1 : h, \mathbf{w}^*)$. Condition (3.4) can be easily verified.

Case 3: Suppose that $B_s = (h-r : h)$, for some $r \geq 1$. By Lemma 3.5, using an argument similar to that in Case 1, $\mathbf{r}_q \sim (h-r : h-1, \mathbf{r}'_q)$ for some tuple $\mathbf{r}'_q \subset \{0, \ldots, h-2\}$. Because $(\mathbf{l}_q, \mathbf{r}_q)$ is symmetric and $\mathbf{l}_q$ and $\mathbf{r}_q$ commute, the index tuple $\mathbf{r}_q$ is symmetric, which implies

$$\mathbf{r}_q \sim (h-r : h-1, \mathbf{r}''_q, h-2 :_\downarrow h-r),$$

for some symmetric tuple $\mathbf{r}''_q \subset \{0, \ldots, h-3\}$. Note that, because of the SIP, $h-1 \notin \mathbf{r}''_q$. For $r > 1$, again by the SIP, $h-2 \notin \mathbf{r}''_q$; for $r = 1$ the same conclusion follows from the symmetry of $\mathbf{r}_q$. Therefore,

$$\begin{aligned}(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) &\sim (\mathbf{l}_q, h-r : h, B_{s-1}, h-r : h-2, B_{s-2}, ..., B_0, h-1, \mathbf{r}'_q) \\ &\sim (h-r : h, \mathbf{l}_q, (B_{s-1}, h-r : h-2, B_{s-2}, ..., B_0), \mathbf{r}''_q, h-1 :_\downarrow h-r).\end{aligned}$$

Observe that, since $\mathbf{l}_q$ and $\mathbf{r}_q$ commute, so do $(h-r : h-1)$ and $\mathbf{l}_q$. Also, since $h-1$ is not in $\mathbf{l}_q$, h commutes with $\mathbf{l}_q$. As $\mathbf{l}_q$ and $\mathbf{r}_q$ are disjoint and $h-1 \in \mathbf{r}_q$, by Lemma 3.4, $h-2, h-1 \notin \mathbf{l}_q$. Thus, by the induction hypothesis, there exist an admissible tuple $\mathbf{w}^*$ relative to $\{0, 1, ..., h-2\}$ and an index tuple $\mathbf{t}^*$ with indices from $\{0, 1, ..., h-3\}$ such that

$$(\mathbf{l}_q, (B_{s-1}, h-r : h-2, B_{s-2}, ..., B_0), \mathbf{r}''_q) \sim (rev(\mathbf{t}^*), \mathbf{w}^*, \mathbf{r}^*_w, \mathbf{t}^*)$$

and $(\mathbf{l}_q, \mathbf{r}''_q) \sim (rev(\mathbf{t}^*), \mathbf{r}^*_w, \mathbf{t}^*)$, where $\mathbf{r}^*_w$ is the symmetric complement of $\mathbf{w}^*$. Then,

$$\begin{aligned}(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) &\sim (h-r : h, rev(\mathbf{t}^*), \mathbf{w}^*, \mathbf{r}^*_w, \mathbf{t}^*, h-1 :_\downarrow h-r) \\ &\sim (h-r : h-2, rev(\mathbf{t}^*), (h-1 : h, \mathbf{w}^*), (\mathbf{r}^*_w, h-1), \mathbf{t}^*, h-2 :_\downarrow h-r),\end{aligned}$$

and (3.3) holds with $\mathbf{t} = (\mathbf{t}^*, h-2 :_\downarrow h-r)$, $\mathbf{r}_w = (h-1, \mathbf{r}^*_w)$ and $\mathbf{w} = (h-1 : h, \mathbf{w}^*)$. Condition (3.4) can be easily verified. □

EXAMPLE 3.16. *Consider the tuples $\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q$ given in Example 3.14. We showed that*

$$(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q) \sim ((rev(0, 2, 1, 4, 3), (5, 1), (6, 3 : 5, 2, 0 : 1), 3, (0, 2, 1, 4, 3)).$$

We also have

$$((5, 1), (6, 3 : 5, 2, 0 : 1), 3) \sim ((5 : 6, 3 : 5, 1 : 2, 0 : 1), 3) \sim ((5 : 6, 3 : 4, 1 : 2, 0), (5, 1, 3)).$$

Thus,

$$(\mathbf{l}_q,\mathbf{q},\mathbf{r}_q)\sim((rev(0,2,1,4,3),(5:6,3:4,1:2,0),(5,1,3),(0,2,1,4,3)).$$

Note that $(5:6,3:4,1:2,0)$ *is an admissible index tuple and* $(5,3,1)$ *is the corresponding symmetric complement.*

The next theorem is the main result of this section and provides a full characterization of the symmetric tuples $(\mathbf{l}_q,\mathbf{q},\mathbf{r}_q)$ satisfying the SIP, with $(\mathbf{l}_q,\mathbf{r}_q)$ symmetric, in terms of admissible tuples.

THEOREM 3.17. *Let* $\mathbf{q}$ *be a permutation of* $\{0,1,\ldots,h\}$, $h\geq 0$, *and* $\mathbf{l}_q,\mathbf{r}_q$ *be index tuples with indices from* $\{0,1,...,h-1\}$ *such that* $(\mathbf{l}_q,\mathbf{q},\mathbf{r}_q)$ *satisfies the SIP. Then,* $(\mathbf{l}_q,\mathbf{q},\mathbf{r}_q)$ *is a symmetric tuple, with* $(\mathbf{l}_q,\mathbf{r}_q)$ *symmetric, if and only if there exist an admissible tuple* $\mathbf{w}$ *relative to* $\{0,1,..,h\}$ *and a tuple* $\mathbf{t}$ *with indices from* $\{0,1,...,h-1\}$ *such that* $(\mathbf{l}_q,\mathbf{q},\mathbf{r}_q)\sim(rev(\mathbf{t}),\mathbf{w},\mathbf{r}_w,\mathbf{t})$ *and* $(\mathbf{l}_q,\mathbf{r}_q)\sim(rev(\mathbf{t}),\mathbf{r}_w,\mathbf{t})$, *where* $\mathbf{r}_w$ *is the symmetric complement of* $\mathbf{w}$.

Proof. Assume that $(\mathbf{l}_q,\mathbf{q},\mathbf{r}_q)$ is a symmetric tuple, with $(\mathbf{l}_q,\mathbf{r}_q)$ symmetric. Then, the claim follows from Lemmas 3.13 and 3.15.

The converse follows from the fact that, by Lemma 3.11, $(\mathbf{w},\mathbf{r}_w)$ and $\mathbf{r}_w$ are symmetric. □

Taking into account the previous theorem, to obtain all possible symmetric tuples $(\mathbf{l}_q,\mathbf{q},\mathbf{r}_q)$ satisfying the SIP and such that $(\mathbf{l}_q,\mathbf{r}_q)$ is symmetric, it is enough to consider all admissible tuples $\mathbf{w}$ and all tuples $\mathbf{t}$ such that $(rev(\mathbf{t}),\mathbf{w},\mathbf{r}_w,\mathbf{t})$ satisfies the SIP, where $\mathbf{r}_w$ is the symmetric complement of $\mathbf{w}$. Next we characterize all tuples $\mathbf{t}$ with such property.

DEFINITION 3.18. *Let* $\mathbf{w}$ *be an admissible tuple relative to* $\{0,1,\ldots,h\}$, $h\geq 0$, *and* $\mathbf{r}_w$ *be the symmetric complement of* $\mathbf{w}$. *We say that a tuple* $\mathbf{t}$ *with indices from* $\{0,\ldots,h-1\}$ *is* $\mathbf{w}$-compatible *if, for any index* i *occurring in both* $\mathbf{r}_w$ *and* $\mathbf{t}$, *the subtuple of* $\mathbf{t}$ *with indices from* $\{i,i+1\}$ *starts with* $i+1$.

LEMMA 3.19. *Let* $\mathbf{w}$ *be an admissible tuple relative to* $\{0,\ldots,h\}$, $h\geq 0$. *Let* $\mathbf{r}_w$ *be the symmetric complement of* $\mathbf{w}$ *and* $\mathbf{t}$ *be a tuple with indices from* $\{0,\ldots,h-1\}$. *Then,* $(rev(\mathbf{t}),\mathbf{w},\mathbf{r}_w,\mathbf{t})$ *satisfies the SIP if and only if*

i) $\mathbf{t}$ *satisfies the SIP*

ii) $\mathbf{t}$ *is* $\mathbf{w}$*-compatible.*

Proof. Assume that $(rev(\mathbf{t}),\mathbf{w},\mathbf{r}_w,\mathbf{t})$ satisfies the SIP. By Remark 2.17, condition i) holds. Condition ii) follows because, by definition of $\mathbf{r}_w$, for any index i in $\mathbf{r}_w$, the subtuple of $\mathbf{r}_w$ with indices from $\{i,i+1\}$ finishes with i.

Assume that $\mathbf{t}$ satisfies the SIP and is $\mathbf{w}$-compatible. Since $(\mathbf{w},\mathbf{r}_w)$ and $\mathbf{t}$ satisfy the SIP, it is enough to check that between any two indices equal to i, one appearing in $(\mathbf{w},\mathbf{r}_w)$ and the other in $\mathbf{t}$, there is an index $i+1$. But this follows from ii) if i is in $\mathbf{r}_w$. If $i<h$ is in $\mathbf{w}$ but not in $\mathbf{r}_w$ then $i+1$ is in $\mathbf{r}_w$, by definition of $\mathbf{r}_w$, and the result follows. □

Note that if $(rev(\mathbf{t}),\mathbf{w},\mathbf{r}_w,\mathbf{t})$ satisfies the SIP, because $h-1$ is in $\mathbf{r}_w$ and h is neither in $\mathbf{t}$ nor in $\mathbf{r}_w$, then $h-1$ is not in $\mathbf{t}$.

The next example describes, up to equivalence, all tuples $\mathbf{t}$ such that $(rev(\mathbf{t}),\mathbf{w},\mathbf{r}_w,\mathbf{t})$ satisfies the SIP, for a given admissible tuple $\mathbf{w}$.

EXAMPLE 3.20. *Consider the admissible tuple* $\mathbf{w}=(5:6,3:4,0:2)$ *and its symmetric complement* $\mathbf{r}_w=(5,3,0,1,0)$. *We describe, up to equivalence, the tuples* $\mathbf{t}$ *with indices from* $\{0,\ldots,5\}$ *such that* $(rev(\mathbf{t}),\mathbf{w},\mathbf{r}_w,\mathbf{t})$ *satisfies the SIP. Note that* $5\notin\mathbf{t}$.

Suppose that $4 \in \mathbf{t}$. Then, because $5 \notin \mathbf{t}$ and $\mathbf{t}$ satisfies the SIP, 4 occurs exactly once. Thus the subtuple of $\mathbf{t}$ with indices from $\{4\}$ is of the form

$$(4).$$

Suppose that $3 \in \mathbf{t}$. Then, because $3 \in \mathbf{r}_w$, by Lemma 3.19, $4 \in \mathbf{t}$ and occurs before the first occurrence of 3. Thus, the subtuple of $\mathbf{t}$ with indices from $\{3,4\}$ is of the form

$$(4,3).$$

Suppose that $2 \in \mathbf{t}$. If $3 \in \mathbf{t}$, by the previous case, the subtuple of $\mathbf{t}$ with indices from $\{2,3,4\}$ has one of the following forms:

$$(2,4,3), \qquad (2,4,3,2), \qquad (4,3,2).$$

If $3 \notin \mathbf{t}$, the subtuple with indices from $\{2,3,4\}$ has one of the following forms:

$$(2), \qquad (2,4).$$

Suppose that $1 \in \mathbf{t}$. Then, by Lemma 3.19, $2 \in \mathbf{t}$ occurs before the first occurrence of 1. Thus, the subtuple of $\mathbf{t}$ with indices from $\{1,2,3,4\}$ has one of the following forms:

$$\begin{array}{llll} (2,1,4,3), & (2,1,4,3,2), & (2,1,4,3,2,1), & \\ (2,4,3,2,1), & (4,3,2,1), & (2,1), & (2,1,4). \end{array}$$

Finally, suppose that $0 \in \mathbf{t}$. Then, by Lemma 3.19, $1 \in \mathbf{t}$ occurs before the first occurrence of 0. Thus, the subtuple of $\mathbf{t}$ with indices from $\{0,1,2,3,4\}$ has one of the following forms:

$$\begin{array}{lll} (2,1,0,4,3), & (2,1,0,4,3,2), & (2,1,0,4,3,2,1), \\ (2,1,4,3,2,1,0), & (2,1,0,4,3,2,1,0), & (2,4,3,2,1,0), \\ (4,3,2,1,0), & (2,1,0), & (2,1,4,0). \end{array}$$

The twenty three displayed tuples are all possible tuples $\mathbf{t}$, up to equivalence, such that $(rev(t), \mathbf{w}, \mathbf{r}_w, \mathbf{t})$ satisfies the SIP.

4. Fiedler pencils with repetitions . Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree k as in (1.1). The family of *Fiedler pencils with repetition* (FPR) associated with $P(\lambda)$ was defined in [25]. In this paper, we describe the FPR that are symmetric when $P(\lambda)$ is. Before introducing this definition, we consider the elementary matrices used in their construction.

4.1. The matrices M_i. We start by defining the matrices $M_i(P)$, depending on the coefficients of the matrix polynomial $P(\lambda)$, which appear as factors of the coefficients of a FPR. These matrices $M_i(P)$ are presented as block matrices partitioned into $k \times k$ blocks of size $n \times n$. Unless the context makes it ambiguous, we will denote these matrices by M_i instead of $M_i(P)$.

Define

$$M_0 := \left[\begin{array}{c|c} I_{(k-1)n} & 0 \\ \hline 0 & -A_0 \end{array} \right], \quad M_{-k} := \left[\begin{array}{c|c} A_k & 0 \\ \hline 0 & I_{(k-1)n} \end{array} \right],$$

and

$$M_i := \left[\begin{array}{c|cc|c} I_{(k-i-1)n} & 0 & 0 & 0 \\ \hline 0 & -A_i & I_n & 0 \\ 0 & I_n & 0 & 0 \\ \hline 0 & 0 & 0 & I_{(i-1)n} \end{array}\right], \quad i = 1, \ldots, k-1. \tag{4.1}$$

The matrices M_i in (4.1) are always invertible and their inverses are given by

$$M_{-i} := M_i^{-1} = \left[\begin{array}{c|cc|c} I_{(k-i-1)n} & 0 & 0 & 0 \\ \hline 0 & 0 & I_n & 0 \\ 0 & I_n & A_i & 0 \\ \hline 0 & 0 & 0 & I_{(i-1)n} \end{array}\right].$$

The matrices M_0 and M_{-k} are invertible if and only if A_0 and A_k, respectively, are.

Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple with indices from $\{0, 1, ..., k-1, -1, ..., -k\}$. We denote $M_{\mathbf{t}} := M_{i_1} M_{i_2} \cdots M_{i_r}$. If $\mathbf{t}$ is empty, then $M_{\mathbf{t}} = I_{kn}$. We also use the following notation: $revtr(M_{\mathbf{t}}) = M_{i_1}^T \cdots M_{i_r}^T$. Note that, if $\mathbf{t}$ is symmetric, then $M_{\mathbf{t}}^T = revtr(M_{\mathbf{t}})$.

In [25] it was proven that if $\mathbf{t}$ is an index tuple from $\{0, ..., k-1\}$ (resp. $\{-k, ..., -1\}$) satisfying the SIP then all the $n \times n$ blocks in $M_{\mathbf{t}}$ are of the form 0, I and $-A_i$ (resp. 0, I and A_i). It is also noteworthy that, when performing the products in $M_{\mathbf{t}}$, no cancellations occur.

Remark 4.1. *It is easy to check that the commutativity relations*

$$M_i(P) M_j(P) = M_j(P) M_i(P), \quad \textit{for any } P(\lambda), \tag{4.2}$$

hold if and only if $||i| - |j|| \neq 1$.

Lemma 4.2. *Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be two index tuples with the same indices from either $\{0, 1, \ldots, k-1\}$ or $\{-k, \ldots, -1\}$. Assume that $\mathbf{t}_1$ and $\mathbf{t}_2$ satisfy the SIP. Then,*

i) If $\mathbf{t}_1$ is equivalent to $\mathbf{t}_2$ then $M_{\mathbf{t}_1}(P) = M_{\mathbf{t}_2}(P)$ for any matrix polynomial $P(\lambda)$ of the form (1.1);

ii) If $M_{\mathbf{t}_1}(P) = M_{\mathbf{t}_2}(P)$ for some matrix polynomial $P(\lambda)$ of the form (1.1) with A_0 nonsingular and $A_i \neq -I_n$, for $i = 0, \ldots, k$, then $\mathbf{t}_1$ is equivalent to $\mathbf{t}_2$.

Proof. We consider the case when $\mathbf{t}_1$ and $\mathbf{t}_2$ have indices from $\{0, 1, \ldots, k-1\}$. The proof is similar if the indices are from $\{-k, \ldots, -1\}$.

By Remark 4.1, the matrices M_i and M_j commute for any matrix polynomial $P(\lambda)$ if and only if the indices i and j commute. Thus, if $\mathbf{t}_1 \sim \mathbf{t}_2$, then $M_{\mathbf{t}_1}(P) = M_{\mathbf{t}_2}(P)$ for any matrix polynomial $P(\lambda)$ and i) follows.

Assume now that $\mathbf{t}_1$ and $\mathbf{t}_2$ are not equivalent in order to prove ii). Since $\mathbf{t}_1$ and $\mathbf{t}_2$ satisfy the SIP, they are equivalent to tuples in column standard form, which also satisfy the SIP. Let $csf(\mathbf{t}_1) = (B_{m_1}, ..., B_1, B_0)$ and $csf(\mathbf{t}_2) = (\tilde{B}_{m_2}, ..., \tilde{B}_1, \tilde{B}_0)$. If B_{m_1} and $\tilde{B}_{m_2}$ are distinct, let $r = 0$. Otherwise, let r be the largest positive integer such that $B_{m_1-i+1} = \tilde{B}_{m_2-i+1}$, $i = 1, \ldots, r$. Since $\mathbf{t}_1$ and $\mathbf{t}_2$ have the same indices, we deduce that $(B_{m_1-r}, ..., B_0)$ and $(\tilde{B}_{m_2-r}, ..., \tilde{B}_0)$ have the same indices as well. By the SIP, the largest index in a tuple occurs exactly once, and, by definition of column standard form, it appears in the first string (counting from left to right). Thus, $B_{m_1-r} = (a : b)$ and $\tilde{B}_{m_2-r} = (a' : b)$ for some a, a', and b with $a \neq a'$. Since any index in $(B_{m_1-r-1},\ldots,B_0)$ is smaller than b, we have that $M_{B_{m_1-r-1},\ldots,B_0}(P)$ is of the form

$$\begin{bmatrix} I_{n(k-b)} & 0 \\ 0 & \star \end{bmatrix}, \tag{4.3}$$

where $\star$ denotes unspecified entries. On the other hand, a calculation shows that

$$M_{B_{m_1-r}}(P) = \left[\begin{array}{c|cc|c} I_{n(k-b-1)} & 0 & 0 & 0 \\ \hline 0 & -A_b & \star & 0 \\ 0 & -A_{b-1} & \star & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & -A_a & \star & 0 \\ 0 & I_n & \star & 0 \\ \hline 0 & 0 & 0 & \star \end{array}\right] \tag{4.4}$$

if $a \neq 0$, and

$$M_{B_{m_1-r}}(P) = \left[\begin{array}{c|cc} I_{n(k-b-1)} & 0 & 0 \\ \hline 0 & -A_b & \star \\ 0 & -A_{b-1} & \star \\ \vdots & \vdots & \vdots \\ 0 & -A_1 & \star \\ 0 & -A_0 & \star \end{array}\right], \tag{4.5}$$

if $a = 0$. Therefore, taking into account (4.3) and the form of $M_{B_{m_1-r}}(P)$, it follows that the matrix $M_1 := M_{B_{m_1-r},\ldots,B_0}(P)$ is still of the form (4.4) if $a \neq 0$ and of the form (4.5) if $a = 0$. A similar form can be obtained for $M_2 := M_{\tilde{B}_{m_2-r},\ldots,\tilde{B}_0}(P)$. Since $a \neq a'$, we deduce that $M_{B_{m_1-r},\ldots,B_0}(P) \neq M_{\tilde{B}_{m_2-r},\ldots,\tilde{B}_0}(P)$ as long as $P(\lambda)$ is a matrix polynomial of the form (1.1) with $A_0 \neq 0$ and all its coefficient matrices different from $-I_n$.

Clearly, if $r = 0$, we have that $M_{\mathbf{t}_1}(P) \neq M_{\mathbf{t}_2}(P)$ for some matrix polynomial $P(\lambda)$. Now suppose that $r > 0$. If $P(\lambda)$ is such that A_0 is nonsingular and $M_{B_{m_1-r},\ldots,B_0}(P) \neq M_{\tilde{B}_{m_2-r},\ldots,\tilde{B}_0}(P)$ then, since $M_{B_{m_1},\ldots,B_{m_1-r+1}}(P)(= M_{\tilde{B}_{m_2},\ldots,\tilde{B}_{m_2-r+1}}(P))$ is nonsingular, we have that $M_{\mathbf{t}_1}(P) \neq M_{\mathbf{t}_2}(P)$ for that $P(\lambda)$. Thus, in any case ii) follows. □

4.2. Definition of FPR. We now define the family of Fiedler pencils with repetition, from which we construct the new structured linearizations.

DEFINITION 4.3. *(FPR). Let $P(\lambda)$ be a matrix polynomial of degree k, as in (1.1). Let $h \in \{0,1,\ldots,k-1\}$. Let $\mathbf{q}$ and $\mathbf{z}$ be permutations of $\{0,1,\ldots,h\}$ and $\{-k,-k+1,\ldots,-h-1\}$, respectively. Let $\mathbf{l}_q$ and $\mathbf{r}_q$ be index tuples from $\{0,1,\ldots,h-1\}$ such that $(\mathbf{l}_q,\mathbf{q},\mathbf{r}_q)$ satisfies the SIP. Let $\mathbf{l}_z$ and $\mathbf{r}_z$ be index tuples from $\{-k,-k+1,\ldots,-h-2\}$ such that $(\mathbf{l}_z,\mathbf{z},\mathbf{r}_z)$ satisfies the SIP. Then, the pencil*

$$L(\lambda) = \lambda M_{\mathbf{l}_q,\mathbf{l}_z,\mathbf{z},\mathbf{r}_z,\mathbf{r}_q} - M_{\mathbf{l}_q,\mathbf{l}_z,\mathbf{q},\mathbf{r}_z,\mathbf{r}_q} \tag{4.6}$$

is called a Fiedler pencil with repetition (FPR) *associated with $P(\lambda)$.*

When convenient and in order to make explicit the dependence of $L(\lambda)$ on the matrix polynomial $P(\lambda)$, for fixed tuples $\mathbf{l}_q,\mathbf{q},\mathbf{r}_q,\mathbf{l}_z,\mathbf{z},\mathbf{r}_z$, we denote by $L_P(\lambda)$ the pencil of the form (4.6), with the blocks A_i being the coefficients of $P(\lambda)$, where $P(\lambda)$ is an arbitrary matrix polynomial of degree k.

We observe that $M_{\mathbf{l}_q}$ and $M_{\mathbf{r}_q}$ commute with each factor in $M_{\mathbf{l}_z} M_{\mathbf{z}} M_{\mathbf{r}_z}$. Analogously, $M_{\mathbf{l}_z}$ and $M_{\mathbf{r}_z}$ commute with each factor in $M_{\mathbf{l}_q} M_{\mathbf{q}} M_{\mathbf{r}_q}$.

A FPR as in (4.6) can be expressed as $M_{\mathbf{l}_q,\mathbf{l}_z}(\lambda M_{\mathbf{z}} - M_{\mathbf{q}}) M_{\mathbf{r}_q,\mathbf{r}_z}$. The pencil $\lambda M_{\mathbf{z}} - M_{\mathbf{q}}$ is a generalized Fiedler pencil [1, 2], which is known to be a strong linearization of $P(\lambda)$ [5]. Therefore, we have the following result.

LEMMA 4.4. *Let $P(\lambda)$ be a matrix polynomial and $L_P(\lambda)$ be a FPR as in (4.6). Then, $L_P(\lambda)$ is a strong linearization of $P(\lambda)$, unless one of the following conditions holds:*

i) 0 is an index in $\mathbf{l}_q$ or $\mathbf{r}_q$ and A_0 is singular;

ii) $-k$ is an index in $\mathbf{l}_z$ or $\mathbf{r}_z$ and A_k is singular.

Thus, in order to obtain the symmetric linearizations from FPR we will assume that none of the conditions i) and ii) in Lemma 4.5 hold.

DEFINITION 4.5. *Let $P(\lambda)$ be a matrix polynomial of degree k, as in (1.1), and $L_P(\lambda)$ be a FPR as in (4.6). We say that $L_P(\lambda)$ satisfies the* nonsingularity conditions *if neither of the conditions i) and ii) in Lemma 4.5 holds.*

The blocks that appear in the coefficients of the pencil (4.6) are of the form 0, I_n, and $-A_i$ for some i's [25].

We finish this section by observing that in [25] the coefficients of the FPR are products of the matrices RM_iR, instead of M_i, where R is the $nk \times nk$ matrix

$$R := \begin{bmatrix} 0 & & I_n \\ & \iddots & \\ I_n & & 0 \end{bmatrix}. \tag{4.7}$$

Therefore, if the linearizations in Definition 4.4 are multiplied on the left and on the right by the matrix R, the linearizations constructed in [25] are obtained.

5. Symmetric Linearizations. In Theorem 5.3 in this section we characterize all FPR that are symmetric when the matrix polynomial $P(\lambda)$ of degree k is, which we prove to be equivalent to the characterization of all FPR that are block-symmetric for every matrix polynomial $P(\lambda)$.

We observe that an analog of Theorem 5.3 holds in the Hermitian case. Namely, if $P(\lambda)$ is a Hermitian matrix polynomial of degree k of the form (1.1), then the pencil $P(\lambda)$ given in (5.1) is a Hermitian strong linearization of $P(\lambda)$, provided that $L_P(\lambda)$ satisfies the nonsingularity conditions. The proof of this claim is similar to the one of Theorem 5.3, noting that a result analog to Lemma 5.2 holds in the Hermitian case, that is, if $\mathbf{t}$ is a tuple as in the lemma, then $M_{\mathbf{t}}$ is Hermitian for any Hermitian $P(\lambda)$ of degree k if and only if $\mathbf{t}$ is symmetric.

Recall that a matrix polynomial $P(\lambda)$ as in (1.1) is symmetric if $A_i^T = A_i$, $i = 0, 1, \ldots, k$. Thus, when $P(\lambda)$ is symmetric, the matrices M_i and M_{-i} defined in Section 4 are symmetric for $i = 0, 1, \ldots, k$.

We next present a lemma which is crucial in the proof of our main result. Recall the notation introduced in Section 4.

LEMMA 5.1. *Let $\mathbf{t}$ be a tuple satisfying the SIP with indices from either $\{0, 1, \ldots, k-1\}$ or $\{-k, \ldots, -1\}$. Then, $M_{\mathbf{t}}(P)$ is symmetric for any symmetric matrix polynomial $P(\lambda)$ of degree k if and only if $\mathbf{t}$ is symmetric.*

Proof. Assume that $\mathbf{t}$ is symmetric and $P(\lambda)$ is symmetric. By Lemma 4.2, $M_{\mathbf{t}}(P) = M_{rev(\mathbf{t})}(P)$, which implies $M_{\mathbf{t}}^T(P) = revtr(M_{rev(\mathbf{t})}(P)) = revtr(M_{\mathbf{t}}(P)) = M_{\mathbf{t}}(P)$, where the last equality follows from the fact that $P(\lambda)$ is symmetric.

Assume now that $M_{\mathbf{t}}(P)$ is symmetric for any symmetric matrix polynomial $P(\lambda)$. Then, $M_{\mathbf{t}}(P) = M_{\mathbf{t}}^T(P) = revtr(M_{rev(\mathbf{t})}(P)) = M_{rev(\mathbf{t})}(P)$. Again, by Lemma 4.2 again, the result follows. ☐

We now state our main result. It characterizes all the FPR that are symmetric when $P(\lambda)$ is.

THEOREM 5.2. *A pencil $L_P(\lambda)$ of the form (4.6) is symmetric for any symmetric matrix polynomial $P(\lambda)$ as in (1.1) if and only if it can be expressed as*

$$L_P(\lambda) = \lambda M_{rev(\mathbf{t}_w),rev(\mathbf{t}_z),\mathbf{z},\mathbf{r}_z,\mathbf{t}_z,\mathbf{r}_w,\mathbf{t}_w} - M_{rev(\mathbf{t}_w),rev(\mathbf{t}_z),\mathbf{w},\mathbf{r}_z,\mathbf{t}_z,\mathbf{r}_w,\mathbf{t}_w}, \tag{5.1}$$

where $\mathbf{w}$ and $\mathbf{w}'$ are admissible tuples relative to $\{0,\ldots,h\}$ and $\{0,\ldots,k-h-1\}$, respectively, $\mathbf{r}_w, \mathbf{r}_{w'}$ are the symmetric complements of $\mathbf{w}$ and $\mathbf{w}'$, respectively, $\mathbf{t}_w \subset \{0,\ldots,h-1\}$ and $\mathbf{t}_{w'} \subset \{0,\ldots,k-h-2\}$ are index tuples satisfying the SIP and such that $\mathbf{t}_w$ is $\mathbf{w}$-compatible and $\mathbf{t}_{w'}$ is $\mathbf{w}'$-compatible, $\mathbf{z} = -k + \mathbf{w}'$, $\mathbf{r}_z = -k + \mathbf{r}_{w'}$ and $\mathbf{t}_z = -k + \mathbf{t}_{w'}$.

Proof. Consider the FPR $L_P(\lambda)$ given in (4.6), associated with the matrix polynomial $P(\lambda)$, as in (1.1). Then, $L_P(\lambda)$ is symmetric if and only if $M_{\mathbf{l}_q,\mathbf{l}_z,\mathbf{z},\mathbf{r}_z,\mathbf{r}_q}$ and $M_{\mathbf{l}_q,\mathbf{l}_z,\mathbf{q},\mathbf{r}_z,\mathbf{r}_q}$ are symmetric or, equivalently, if the tuples

$$M_{\mathbf{l}_q,\mathbf{r}_q}, \quad M_{\mathbf{l}_z,\mathbf{z},\mathbf{r}_z}, \quad M_{\mathbf{l}_z,\mathbf{r}_z}, \quad M_{\mathbf{l}_q,\mathbf{q},\mathbf{r}_q}$$

are symmetric. Taking into account Lemma 5.2, it follows that $L_P(\lambda)$ is symmetric for any symmetric $P(\lambda)$ if and only if $(\mathbf{l}_q,\mathbf{q},\mathbf{r}_q)$, $(\mathbf{l}_q,\mathbf{r}_q)$, $(\mathbf{l}_z,\mathbf{z},\mathbf{r}_z)$, and $(\mathbf{l}_z,\mathbf{r}_z)$ are symmetric. Now the result follows from Remark 2.17, Theorem 3.17 and Lemma 3.19. ☐

Taking into account Theorem 5.3 and Lemma 4.5, we obtain the next corollary, which gives symmetric strong linearizations of symmetric matrix polynomials.

COROLLARY 5.3. *Let $P(\lambda)$ be a symmetric matrix polynomial as in (1.1) and $L_P(\lambda)$ be a pencil of the form (5.1), as described in Theorem 5.3. If $L_P(\lambda)$ satisfies the nonsingularity conditions then $L_P(\lambda)$ is a symmetric strong linearization of $P(\lambda)$.*

REMARK 5.4. *When k is even and both coefficients A_0 and A_k of $P(\lambda)$ are singular, no pencil $L_P(\lambda)$ given in Theorem 5.3 satisfies the nonsingularity conditions, since h and $k-h-1$ cannot be both even and, therefore, either $\mathbf{w}$ or $\mathbf{w}'$ has odd index, which implies that either $-k$ is in $\mathbf{r}_z$ or 0 is in $\mathbf{r}_w$. Thus, in this case the theorem does not give symmetric FPR that are strong linearizations of $P(\lambda)$. If k is even and not both A_0 and A_k are singular, Theorem 5.3 produces symmetric strong linearizations. In fact, if A_0 is singular and A_k is nonsingular, by choosing h even, $\mathbf{w}$ of index 0 and $\mathbf{t}_w$ not containing 0, the pencil (5.1) satisfies the nonsingularity conditions. If A_0 is nonsingular and A_k is singular, by choosing h odd (so that $k-h-1$ is even), $\mathbf{w}'$ of index 0 and $\mathbf{t}_{w'}$ not containing 0, the pencil (5.1) satisfies the nonsingularity conditions. When k is odd Theorem 5.3 produces symmetric strong linearizations for any symmetric $P(\lambda)$ of degree k.*

We now give an alternative statement to Theorem 5.3 in terms of block-symmetry, which will be a key result for the construction of skew-symmetric and T-alternating strong linearizations in the second part of this paper. We first recall the definition of block-symmetry.

Let $A \in M_{nk}$ be a $k \times k$ block-matrix consisting of block entries A_{ij} of size $n \times n$. We say that A is block-symmetric if $A_{ij} = A_{ji}$ for all $i, j = 1, \ldots, k$.

LEMMA 5.5. *Let $\mathbf{s}$ be a tuple satisfying the SIP with indices from either $\{0,1,\ldots,k-1\}$ or $\{-k,\ldots,-1\}$. Then, $M_{\mathbf{s}}(P)$ is symmetric for any symmetric $P(\lambda)$ of degree k if and only if $M_{\mathbf{s}}(P)$ is block-symmetric for any $P(\lambda)$ of degree k.*

Proof. The claim can be easily verified taking into account that, since $\mathbf{s}$ satisfies the SIP, all blocks in $M_{\mathbf{s}}(P)$ are of the form 0, I_n, or $\pm A_i$. $\square$

We then have the following consequence of Theorem 5.3 and Lemma 5.1, which characterizes all FPR that are block-symmetric for any $P(\lambda)$.

COROLLARY 5.6. *A pencil $L_P(\lambda)$ of the form (4.6) is block-symmetric for any matrix polynomial $P(\lambda)$ as in (1.1) if and only if it can be expressed as a pencil of the form (5.1) as described in Theorem 5.3.*

We observe that if $P(\lambda)$ is a matrix polynomial whose coefficient matrices are in an arbitrary ring with a multiplicative identity element, not necessarily $\mathbb{F}^{n\times n}$, with $\mathbb{F}$ a field, and we define a FPR as before but now considering the blocks of the matrices M_i in such a ring, the "if" claim in the previous corollary, as well as in Theorem 5.3, will hold. In fact, all the results in Sections 2 and 3 are on index tuples and, therefore, are independent of $P(\lambda)$. Also, the "if" claim in Lemma 5.2 holds when we consider such a ring, as claim i) in Lemma 4.2 holds.

Next we show that the pencils in the standard basis for $\mathbb{DL}(P)$ are block-symmetric FPR. This result was proven in [25, Corollary 1]; however, since the notation and approach used in that paper is slightly different than ours, we include here a description for clarification.

Suppose that $P(\lambda)$ is given by (1.1). For $j = 1, \dots, k$, let

$$L_j(P) = \begin{bmatrix} 0 & 0 & \cdots & A_k \\ 0 & & \iddots & A_{k-1} \\ \vdots & \iddots & \iddots & \vdots \\ A_k & A_{k-1} & \dots & A_{k-j+1} \end{bmatrix}, \quad U_j(P) = \begin{bmatrix} A_{j-1} & \dots & A_1 & A_0 \\ \vdots & \iddots & \iddots & \vdots \\ A_1 & \iddots & & 0 \\ A_0 & \cdots & 0 & 0 \end{bmatrix}$$

Let $X_m := L_m(P) \oplus -U_{k-m}(P)$, for $m = 0, \dots, k$. Then, the so-called *standard basis for* $\mathbb{DL}(P)$ is $\{D_1(\lambda, P), \dots, D_k(\lambda, P)\}$, with

$$D_m(\lambda, P) = \lambda X_m - X_{m-1}, \quad \text{for } m = 1, \dots, k.$$

Note that $D_m(\lambda, P)$ is the pencil in $\mathbb{DL}(P)$ with ansatz vector e_m, where $e_i \in \mathbb{F}^k$ denotes the ith vector of the standard basis for $\mathbb{F}^k$.

LEMMA 5.7. *Let $P(\lambda)$ be a matrix polynomial of degree k as in* (1.1). *Then, the pencil $D_m(\lambda, P)$, $m = 1, \dots, k$, is the FPR of the form (5.1), where $\mathbf{w}$ and $k + \mathbf{z}$ are the admissible tuples of index 0 or 1 associated with $k - m$ and $m - 1$, respectively, $\mathbf{r}_w$ and $k + \mathbf{r}_z$ are the symmetric complements of $\mathbf{w}$ and $k + \mathbf{z}$, respectively, and*

$$\mathbf{t}_w = rev\left(0 : k - m - 2, 0 : k - m - 4, \dots, 0 : k - m - 2\left\lfloor \frac{k - m}{2} \right\rfloor\right),$$

$$k + \mathbf{t}_z = rev\left(0 : m - 3, 0 : m - 5, \dots, 0 : m - 1 - 2\left\lfloor \frac{m - 1}{2} \right\rfloor\right).$$

Proof. A computation shows that

$$D_m(\lambda, P) = \lambda M_{(0:k-m)_{rev_c}} M_{-k+(0:m)_{rev_c}} - M_{(0:k-m+1)_{rev_c}} M_{-k+(0:m-1)_{rev_c}}.$$

Now the result follows from Lemma 4.2 and the fact that $(rev(\mathbf{t}_w), \mathbf{w}, \mathbf{r}_w, \mathbf{t}_w) \sim (0 : k-m+1)_{rev_c}$, $(rev(\mathbf{t}_w), \mathbf{r}_w, \mathbf{t}_w) \sim (0 : k-m)_{rev_c}$, $(rev(\mathbf{t}_z), \mathbf{z}, \mathbf{r}_z, \mathbf{t}_z) \sim -k+(0 : m)_{rev_c}$, and $(rev(\mathbf{t}_z), \mathbf{r}_z, \mathbf{t}_z) \sim -k + (0 : m - 1)_{rev_c}$. $\square$

Table 5.1
Example 5.6

$\mathbf{w}$	$\mathbf{r}_w$	$\mathbf{t}_w$
0	$\emptyset$	$\emptyset$
(0:1)	(0)	$\emptyset$
(1:2,0)	(1)	$\emptyset$, (0)
(2:3,0:1)	(2, 0)	$\emptyset$, (1), (1,0)

Table 5.2
Example 5.6

$\mathbf{z}$	$\mathbf{r}_z$	$\mathbf{t}_z$
-4	$\emptyset$	$\emptyset$
(-4:-3)	(-4)	$\emptyset$
(-3:-2, -4)	(-3)	$\emptyset$, (-4)
(-2:-1, -4:-3)	(-2, -4)	$\emptyset$, (-3), (-3, -4)

We finish this section with an application of Theorem 5.3.

Example 5.8. *Let $P(\lambda)$ be a symmetric matrix polynomial of degree $k = 4$. We construct all possible symmetric strong linearizations of $P(\lambda)$ in the family of FPR. We assume that A_0 (resp. A_k) is invertible if* 0 *is an index in $(\mathbf{r}_w, \mathbf{t}_w)$ (resp. $-k$ is an index in $(\mathbf{r}_z, \mathbf{t}_z)$), so that each given pencil satisfies the nonsingularity conditions. Tables 5.1 and 5.2 provide the necessary tuples to construct those pencils.*

Thus, the appropriate combination of the tuples in Tables 5.1 and 5.2 produces, in total, ten symmetric FPR. Note that these pencils are distinct if and only if $A_4 \neq I$ and $A_0 \neq -I$.

The first four linearizations $L_1(\lambda)$, $L_2(\lambda)$, $L_3(\lambda)$, and $L_4(\lambda)$ in our list are those obtained by taking, respectively,

$$\begin{aligned}
&\mathbf{w} = (0), \mathbf{t}_w = \emptyset, \mathbf{z} = (-2:-1, -4:-3), \mathbf{t}_z = (-3, -4),\\
&\mathbf{w} = (0:1), \mathbf{t}_w = \emptyset, \mathbf{z} = (-3:-2, -4), \mathbf{t}_z = (-4),\\
&\mathbf{w} = (1:2, 0), \mathbf{t}_w = (0), \mathbf{z} = (-4:-3), \mathbf{t}_z = \emptyset,\\
&\mathbf{w} = (2:3, 0:1), \mathbf{t}_w = (1, 0), \mathbf{z} = (-4), \mathbf{t}_z = \emptyset,
\end{aligned}$$

and are the pencils in the standard basis of $\mathbb{DL}(P)$. Next we give the explicit expression of the other pencils.

Next we give the explicit expression of these pencils. We first list the four linearizations in the basis of $\mathbb{DL}(P)$ given in [20].

- *Let $\mathbf{w} = (0)$, $\mathbf{t}_w = \emptyset$, $\mathbf{z} = (-2:-1, -4:-3)$, $\mathbf{t}_z = (-3)$. Then, we get*

$$L_5(\lambda) = \lambda \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & 0 & A_4 & A_3 \\ 0 & A_4 & A_3 & A_2 \\ I & A_3 & A_2 & A_1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & A_4 & A_3 & 0 \\ I & A_3 & A_2 & 0 \\ 0 & 0 & 0 & -A_0 \end{bmatrix}.$$

- *Let $\mathbf{w} = (0:1)$, $\mathbf{t}_w = \emptyset$, $\mathbf{z} = (-3:-2, -4)$, $\mathbf{t}_z = \emptyset$. Then, we get*

$$L_6(\lambda) = \lambda \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & A_4 & A_3 & 0 \\ I & A_3 & A_2 & 0 \\ 0 & 0 & 0 & -A_0 \end{bmatrix} - \begin{bmatrix} 0 & I & 0 & 0 \\ I & A_3 & 0 & 0 \\ 0 & 0 & -A_1 & -A_0 \\ 0 & 0 & -A_0 & 0 \end{bmatrix}.$$

- *Let* $\mathbf{w} = (1:2,0)$, $\mathbf{t}_w = \emptyset$, $\mathbf{z} = (-4:-3)$, $\mathbf{t}_z = \emptyset$. *Then, we get*

$$L_7(\lambda) = \lambda \begin{bmatrix} 0 & A_4 & 0 & 0 \\ A_4 & A_3 & 0 & 0 \\ 0 & 0 & -A_1 & I \\ 0 & 0 & I & 0 \end{bmatrix} - \begin{bmatrix} A_4 & 0 & 0 & 0 \\ 0 & -A_2 & -A_1 & I \\ 0 & -A_1 & -A_0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}.$$

- *Let* $\mathbf{w} = (2:3,0:1)$, $\mathbf{t}_w = (1)$, $\mathbf{z} = (-4)$, $\mathbf{t}_z = \emptyset$. *Then, we get*

$$L_8(\lambda) = \lambda \begin{bmatrix} A_4 & 0 & 0 & 0 \\ 0 & -A_2 & -A_1 & I \\ 0 & -A_1 & -A_0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} - \begin{bmatrix} -A_3 & -A_2 & -A_1 & I \\ -A_2 & -A_1 & -A_0 & 0 \\ -A_1 & -A_0 & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix}.$$

- *Let* $\mathbf{w} = (0)$, $\mathbf{t}_w = \emptyset$, $\mathbf{z} = (-2:-1,-4:-3)$, $\mathbf{t}_z = \emptyset$. *Then, we get*

$$L_9(\lambda) = \lambda \begin{bmatrix} 0 & 0 & A_4 & 0 \\ 0 & 0 & 0 & I \\ A_4 & 0 & A_3 & A_2 \\ 0 & I & A_2 & A_1 \end{bmatrix} - \begin{bmatrix} A_4 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & A_2 & 0 \\ 0 & 0 & 0 & -A_0 \end{bmatrix}.$$

- *Let* $\mathbf{w} = (2:3,0:1)$, $\mathbf{t}_w = \emptyset$, $\mathbf{z} = (-4)$, $\mathbf{t}_z = \emptyset$. *Then, we get*

$$L_{10}(\lambda) = \lambda \begin{bmatrix} A_4 & 0 & 0 & 0 \\ 0 & -A_2 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & -A_0 \end{bmatrix} - \begin{bmatrix} -A_3 & -A_2 & I & 0 \\ -A_2 & -A_1 & 0 & -A_0 \\ I & 0 & 0 & 0 \\ 0 & -A_0 & 0 & 0 \end{bmatrix}.$$

6. Conclusions. In this paper we have studied the Fiedler pencils with repetition which are symmetric whenever the matrix polynomial $P(\lambda)$ is. We have characterized all such pencils and have also given sufficient conditions for them to be strong linearizations of $P(\lambda)$. When the matrix polynomial $P(\lambda)$ has degree k and the coefficients of the terms of degree 0 and k are nonsingular, our family is a nontrivial extension of the standard basis of the k-dimensional vector space $\mathbb{DL}(P)$ studied in [15, 20]. It is still an open question the exact number of distinct pencils in our family. However we observe that this number depends on the specific values of the coefficients of the matrix polynomial $P(\lambda)$. We finally note that our family contains symmetric strong linearizations for symmetric singular matrix polynomials when k is odd. Note that while for regular matrix polynomials $P(\lambda)$ almost all pencils in $\mathbb{DL}(P)$ are strong linearizations of $P(\lambda)$, there are no strong linearizations in $\mathbb{DL}(P)$ for singular matrix polynomials $P(\lambda)$ [6].

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