LINEARIZATIONS OF HERMITIAN MATRIX POLYNOMIALS PRESERVING THE SIGN CHARACTERISTIC

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Abstract. The development of strong linearizations preserving whatever structure a matrix polynomial might possess has been a very active area of research in the last years, since such linearizations are the starting point of numerical algorithms for computing eigenvalues of structured matrix polynomials with the properties imposed by the considered structure. In this context, Hermitian matrix polynomials are one of the most important classes of matrix polynomials arising in applications and their real eigenvalues are of great interest. The sign characteristic is a set of signs attached to these real eigenvalues which is crucial for determining the behavior of systems described by Hermitian matrix polynomials and, therefore, it is desirable to develop linearizations that preserve the sign characteristic of these polynomials, but, at present, only one such linearization is known. In this paper, we present a complete characterization of all the Hermitian strong linearizations that preserve the sign characteristic of a given Hermitian matrix polynomial and identify several families of such linearizations that can be constructed very easily from the coefficients of the polynomial.

Key words. congruence, generalized Fiedler pencil, Hermitian matrix polynomial, sign characteristic, linearization, strong linearization.

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1. Introduction.

In this paper, we study matrix polynomials of the type

$$P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i \quad \text{with } A_i = A_i^* \in \mathbb{C}^{n\times n} \text{ for } i = 0, 1, \ldots, k, \tag{1.1}$$

which are known as Hermitian matrix polynomials [16, 17]. In addition, we assume the generic condition that the matrix A_k is nonsingular. Hermitian matrix polynomials arise very often in applications and many different subclasses of them have been studied in the literature (see [2] and the references therein), which are used to model systems with different behaviors. Reference [2] has established an exhaustive classification of numerous types of Hermitian matrix polynomials in terms of the *sign characteristic* of their real eigenvalues, which is a set of signs associated with such eigenvalues (see Section 2) that is crucial for determining the behavior of systems described by Hermitian polynomials.

Sign characteristics are also defined for other classes of structured matrix problems, like Hamiltonian matrices with respect to skew-symmetric bilinear forms or self-adjoint matrices with respect to indefinite inner products. In all the eigenproblems related to these structures, the sign characteristic is essential to understand the striking differences between the behavior of the eigenvalues under structured and unstructured perturbations and, therefore, the drawbacks and difficulties of using numerical algorithms that do not preserve the structures of these problems to compute their eigenvalues [1, 4, 6, 27, 28, 29] (see also Section 3).

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On the other hand, the only reliable method to compute *all* the eigenvalues of a matrix polynomial is via linearizations (see Definition 2.1), which is the method used for instance in MATLAB. In addition, the preservation of structures and the design of structured algorithms require the use of *structured linearizations*. These are two of the reasons that have motivated in the last years an intense activity in the development of new classes of linearizations preserving the different structures that matrix polynomials arising in applications often posses, including the Hermitian structure. In addition, for practical purposes, it is essential that these linearizations are easily constructible from the coefficients of the polynomial. References [3, 7, 10, 11, 12, 19, 21, 22, 32] are a small sample of papers where new classes of linearizations have been presented.

However, as will follow from the discussion in this paper, the sign characteristic of a Hermitian linearization of a Hermitian matrix polynomial $P(\lambda)$ may be different from the sign characteristic of $P(\lambda)$ itself. Therefore, the discussion above makes clear that the development of Hermitian linearizations $L(\lambda)$ of $P(\lambda)$ with the same sign characteristic as $P(\lambda)$ is a very important step towards a complete reliable solution of the Hermitian polynomial eigenvalue problem. If $L(\lambda)$ has the same sign characteristic as $P(\lambda)$, we will say, for short, that $L(\lambda)$ preserves the sign characteristic of $P(\lambda)$. At present, only one Hermitian linearization of $P(\lambda)$ is known to preserve its sign characteristic, and this has been proved only if $P(\lambda)$ has semisimple real eigenvalues [2, Lemma 2.8]. This particular linearization is the last pencil in the standard basis of the famous space of linearizations $\mathbb{DL}(P)$ introduced in [19, 21].

In this paper, we characterize all the Hermitian (strong) linearizations of a Hermitian matrix polynomial $P(\lambda)$ that preserve its sign characteristic in Section 5 (Theorem 5.3) and, based on this characterization, we identify several classes of such linearizations that can be constructed very easily from the coefficients of $P(\lambda)$. In fact, for $k \geq 3$ we construct infinite sets of such linearizations. Some of these linearizations are very well known in the literature and, in these cases, our contribution is to prove for the first time that they preserve the sign characteristic; in other cases the proposed linearizations are new. In Corollary 7.2 we prove that, besides the last pencil in the standard basis of $\mathbb{DL}(P)$, other pencils in that basis also preserve the sign characteristic of $P(\lambda)$. In Section 6 we show that the renowned block-tridiagonal strong linearizations of $P(\lambda)$ introduced in [3] and in [23, 24, 25] also preserve the sign characteristic of $P(\lambda)$. Note that the linearization in [23, 24, 25] is perhaps the simplest Hermitian strong linearization of $P(\lambda)$ for odd degree, as illustrated by the next example, corresponding to degree $k = 7$,

$$\widetilde{L}(\lambda) = \begin{bmatrix} \lambda A_1 + A_0 & \lambda I & & & & & 0 \\ \lambda I & 0 & I & & & & \\ & I & \lambda A_3 + A_2 & \lambda I & & & \\ & & \lambda I & 0 & I & & \\ & & & I & \lambda A_5 + A_4 & \lambda I & \\ & & & & \lambda I & 0 & I \\ 0 & & & & & I & \lambda A_7 + A_6 \end{bmatrix},$$

which allows us to predict the general pattern for any odd degree. Infinitely many other linearizations that preserve the sign characteristic of $P(\lambda)$ are presented in Section 7. They belong to the family of generalized Fiedler pencils with repetition (GFPR), introduced very recently in [10], which extends in a nontrivial way the set of pencils forming the standard basis of $\mathbb{DL}(P)$. Particularly simple examples of these GFPR linearizations are carefully described in Section 8 (see Example 8.2).

The paper is completed with the presentation in Section 2 of a summary of basic concepts including the classical definition of sign characteristic of a Hermitian matrix polynomial with nonsingular leading coefficient, some numerical experiments that motivate this research in Section 3, some needed background on GFPRs in Section 4, and the conclusions in Section 9.

2. Background. In this section, we refresh some basic concepts and introduce some notation that will be used in the rest of the paper.

A matrix polynomial

$$P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i, \quad A_0, \ldots, A_k \in \mathbb{C}^{n\times n}, \tag{2.1}$$

is *regular* if $\det P(\lambda)$ is not identically zero. The most interesting results presented in this manuscript assume that A_k is invertible, which implies that $P(\lambda)$ is regular and has *degree* k. When $A_k = 0$ and $P(\lambda)$ is written as in (2.1), the degree of $P(\lambda)$ is less than k, in which case we say that the *grade* of $P(\lambda)$ is k. The concept of *grade* [13, 26] allows us greater flexibility in dealing with other auxiliary polynomials. One example is the *reversal* of $P(\lambda)$ of grade k defined as

$$(\mathrm{rev}\, P)(\lambda) := \lambda^k P\left(\frac{1}{\lambda}\right) = \sum_{i=0}^{k} \lambda^i A_{k-i}. \tag{2.2}$$

Note that if $A_0 = 0$, then $(\mathrm{rev}\, P)(\lambda)$ has degree less than k, but $(\mathrm{rev}\, P)(\lambda)$ has *grade* k. We define a *matrix pencil* as a matrix polynomial of grade 1, $L(\lambda) = \lambda X + Y$, without requiring explicitly that $X \neq 0$.

By definition [17, Chapter 7], $P(\lambda)$ has an eigenvalue at infinity if $(\mathrm{rev}\, P)(\lambda)$ has an eigenvalue at 0 and the infinite elementary divisors of $P(\lambda)$ are those of $(\mathrm{rev}\, P)(\lambda)$ for the eigenvalue 0. If $P(\lambda)$ is regular, then $P(\lambda)$ has an eigenvalue at infinity if and only if A_k is singular.

We use the classical definitions of linearization and strong linearization [15, 17] of a matrix polynomial. In this paper, I_m denotes the $m \times m$ identity matrix.

Definition 2.1. *A matrix pencil $L(\lambda) = \lambda X + Y$ with $X, Y \in \mathbb{C}^{nk\times nk}$ is a* linearization *of an $n\times n$ matrix polynomial $P(\lambda)$ of grade k if there exist two unimodular $nk \times nk$ matrix polynomials $U(\lambda)$ and $V(\lambda)$, i.e., matrix polynomials with constant nonzero determinant, such that*

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}. \tag{2.3}$$

A linearization $L(\lambda)$ is called a strong linearization *if $(\mathrm{rev}\, L)(\lambda)$ is also a linearization of $(\mathrm{rev}\, P)(\lambda)$.*

According to [13, Theorem 4.1], the linearizations of a regular matrix polynomial $P(\lambda)$ as in (2.1) are precisely those $nk \times nk$ regular pencils that have the same finite elementary divisors as $P(\lambda)$, and the strong linearizations of a regular matrix polynomial $P(\lambda)$ are precisely those $nk \times nk$ regular pencils that have the same finite and infinite elementary divisors as $P(\lambda)$. Therefore, for a regular matrix polynomial $P(\lambda)$ as in (2.1) without infinite eigenvalues, i.e., with A_k nonsingular, its linearizations and strong linearizations are the same.

Two matrix pencils $\lambda X + Y$ and $\lambda\widetilde{X} + \widetilde{Y}$ are *strictly equivalent* if there exist two nonsingular matrices Q and S such that $\lambda X + Y = Q(\lambda\widetilde{X} + \widetilde{Y})S$. From [14, Chapter

XII], we get that given a regular matrix polynomial $P(\lambda)$ and a strong linearization $L(\lambda)$ of $P(\lambda)$, another pencil is a strong linearization of $P(\lambda)$ if and only if it is strictly equivalent to $L(\lambda)$. If, in addition, $P(\lambda)$ has no eigenvalues at infinity and $L(\lambda)$ is a linearization of $P(\lambda)$, then another pencil is a linearization of $P(\lambda)$ if and only if it is strictly equivalent to $L(\lambda)$.

Two pencils $\lambda X + Y$ and $\lambda \widetilde{X} + \widetilde{Y}$ are **congruent* if there exists a nonsingular matrix Q such that $\lambda X + Y = Q(\lambda \widetilde{X} + \widetilde{Y})Q^*$. Clearly, two *congruent pencils are strictly equivalent.

The most relevant linearizations of $n \times n$ matrix polynomials of grade k used in practice are easily described when their coefficients are viewed as block-matrices partitioned into $k \times k$ blocks of size $n \times n$ [3, 7, 10, 11, 12, 17, 19, 21, 22, 32]. The description of some structures of linearizations requires the definition of *block-transposition* and *block-symmetry*. To this purpose, let $H = (H_{ij})$ be a $k \times k$ block-matrix with $H_{ij} \in \mathbb{C}^{n\times n}$. Then the block-transpose of H is $H^{\mathcal{B}} := (H_{ji})$, i.e., $H^{\mathcal{B}}$ has H_{ji} in the block-entry (i, j), and H is block-symmetric if $H = H^{\mathcal{B}}$. Observe that if $H_{ij} = H_{ij}^*$ for all (i, j), then $H = H^{\mathcal{B}}$ implies $H = H^*$.

Note that the linearizations studied in this paper are more naturally written as $L(\lambda) = \lambda L_1 - L_0$, i.e., by extracting a minus sign from the zero-degree coefficient.

Next we focus on Hermitian matrix polynomials and give one of the classical definitions of sign characteristic of $P(\lambda)$ when $P(\lambda) = \sum_{i=0}^{k} A_i\lambda^i$ is Hermitian with nonsingular leading coefficient A_k. Note that, in this case, $P(\lambda)$ is regular and has no infinite elementary divisors. See [16] and [17] for more details on this topic, as well as [30].

We define the $nk \times nk$ matrices

$$C_P := \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & I_n \\ -A_k^{-1}A_0 & -A_k^{-1}A_1 & \cdots & \cdots & -A_k^{-1}A_{k-1} \end{bmatrix} \tag{2.4}$$

and

$$B_P := \begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ A_2 & \vdots & \iddots & \\ \vdots & A_k & & \\ A_k & & & 0 \end{bmatrix}. \tag{2.5}$$

Note that, if $L(\lambda) = \lambda L_1 - L_0$ is a matrix pencil, with L_1 nonsingular, then $C_L = L_1^{-1}L_0$ and $B_L = L_1$.

Since the pencil $\lambda I_{nk} - C_P$ is obtained from the first Frobenius companion form of $P(\lambda)$ [17] by premultiplying it by $I_{n(k-1)} \oplus A_k^{-1}$, it follows that it is a strong linearization of $P(\lambda)$ and preserves the (finite) elementary divisors of $P(\lambda)$.

When $P(\lambda)$ is Hermitian, the sizes of the Jordan blocks of C_P with a nonreal eigenvalue λ_0 (that is, the degrees of the elementary divisors associated with λ_0) are equal to the sizes of the Jordan blocks with eigenvalue $\overline{\lambda_0}$ [16, Proposition 4.2.3]. Thus, we can assume that the Jordan form J of $P(\lambda)$ is a direct sum of Jordan blocks associated with real eigenvalues and blocks of the type $diag(J_r, \overline{J_r})$, where J_r is an

$r \times r$ Jordan block associated with a nonreal eigenvalue λ and $\overline{J_r}$ is a Jordan block of the same size associated with $\overline{\lambda}$.

Following the notation in [16, p. 74], in Theorem 2.2 below $J_r(\lambda)$ denotes the Jordan block associated with the eigenvalue λ of size $r \times r$ if λ is real, and the direct sum of two Jordan blocks of size $r/2 \times r/2$, if λ is not real, in which case the first block corresponds to λ and the second to $\overline{\lambda}$. We also need the sip ("standard involutory permutation" [16, p. 8]) matrix of size $m \times m$, i.e.,

$$\mathcal{R}_m := \begin{bmatrix} 0 & \dots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \dots & 0 \end{bmatrix}. \tag{2.6}$$

We will additionally use the $m \times m$ block-sip matrix with $n \times n$ blocks, i.e.,

$$R_m := \begin{bmatrix} 0 & \dots & I_n \\ \vdots & \iddots & \vdots \\ I_n & \dots & 0 \end{bmatrix}. \tag{2.7}$$

Observe that we denote with a calligraphic font the entrywise sip matrix.

THEOREM 2.2. *Let $P(\lambda)$ be an Hermitian matrix polynomial with nonsingular leading coefficient. Let $\lambda_1, \dots, \lambda_\alpha$ be the real eigenvalues of $P(\lambda)$, and $\lambda_{\alpha+1}, \dots, \lambda_\beta$ be the nonreal eigenvalues of $P(\lambda)$ from the upper half-plane. Then, there exists a nonsingular matrix H such that*

$$J := H^{-1} C_P H = J_{l_1}(\lambda_1) \oplus \dots \oplus J_{l_\alpha}(\lambda_\alpha) \oplus J_{l_{\alpha+1}}(\lambda_{\alpha+1}) \oplus \dots \oplus J_\beta(\lambda_\beta) \tag{2.8}$$

and

$$P_{\epsilon,J} := H^* B_P H = \epsilon_1 \mathcal{R}_{l_1} \oplus \dots \oplus \epsilon_\alpha \mathcal{R}_{l_\alpha} \oplus \mathcal{R}_{l_{\alpha+1}} \oplus \dots \oplus \mathcal{R}_{l_\beta}, \tag{2.9}$$

where $\epsilon = \{\epsilon_1, \dots, \epsilon_\alpha\}$ is an ordered set of signs ± 1. The set ϵ is unique up to permutation of signs corresponding to equal Jordan blocks.

DEFINITION 2.3. *Let $P(\lambda)$ be as in Theorem 2.2. The set $\{\epsilon_1, \dots, \epsilon_\alpha\}$ in that theorem is called the sign characteristic of $P(\lambda)$.*

In the special case in which λ_0 is a simple eigenvalue of a Hermitian matrix polynomial $P(\lambda)$, the sign in the sign characteristic of $P(\lambda)$ corresponding to λ_0 is given by

$$sign(x^* P'(\lambda_0) x), \tag{2.10}$$

where x is an eigenvector of $P(\lambda)$ associated with λ_0 and $P'(\lambda)$ is the first derivative of $P(\lambda)$ [2, 16].

3. Motivation and numerical experiments. As mentioned in the introduction, the sign characteristic is essential in describing the behavior of the real eigenvalues of a Hermitian matrix polynomial under perturbations that preserve the structure. Since the goal of a good linearization is to reflect as fairly as possible the properties of the original matrix polynomial, it is convenient that a linearization of a Hermitian matrix polynomial preserves its sign characteristic. In fact, using Hermitian linearizations that do not preserve the sign characteristic may lead to unreliable results which do not reflect the spectral properties of the original polynomial when the linearizations are perturbed as a consequence, for instance, of backward errors in numerical computations. For example, there exist applications of Hermitian matrix polynomials where the main goal is to determine whether or not all their eigenvalues are real. A nice example is the parameterized quadratic Hermitian matrix polynomial arising in the stability analysis of linear second order gyroscopic systems discussed in [30, Example 1.3] and the references therein. Since the real nature of all the eigenvalues has to be determined numerically, the use of linearizations is natural and the first decision to make is which linearization to use. This section illustrates the importance of using linearizations preserving the sign characteristic with a numerical example, and motivates in this way the results presented in this paper.

Let us consider the following 2×2 quadratic Hermitian matrix polynomial with nonsingular leading coefficient

$$\begin{aligned} P(\lambda) &= \begin{bmatrix} (\lambda+2)(\lambda-1+\delta) & 0 \\ 0 & (2-\lambda)(\lambda-1-\delta) \end{bmatrix} \\ &= \lambda^2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \lambda \begin{bmatrix} 1+\delta & 0 \\ 0 & 3+\delta \end{bmatrix} + \begin{bmatrix} 2(\delta-1) & 0 \\ 0 & -2(\delta+1) \end{bmatrix}, \end{aligned} \tag{3.1}$$

where $\delta > 0$ is a small parameter that in our numerical tests takes the value $\delta = 10^{-7}$. Note that $P(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ has four simple real eigenvalues, $2, -2, 1-\delta, 1+\delta$, two of which, $1+\delta, 1-\delta$, are very close together. Since the eigenvalues of $P(\lambda)$ are simple, using (2.10), we verify that the signs attached to $1+\delta$ and $1-\delta$ are both $+1$. We will analyze in the numerical tests below the behavior under perturbations of the following two Hermitian linearizations of $P(\lambda)$ in (3.1):

$$L(\lambda) = \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1-\delta & 0 \\ 0 & 0 & 0 & -1-\delta \end{bmatrix}, \tag{3.2}$$

$$D_2(\lambda, P) = \lambda \begin{bmatrix} 0 & A_2 \\ A_2 & A_1 \end{bmatrix} + \begin{bmatrix} -A_2 & 0 \\ 0 & A_0 \end{bmatrix}. \tag{3.3}$$

It is well-known that $D_2(\lambda, P)$ preserves the sign characteristic of $P(\lambda)$ [2] (see also Corollary 7.2 in this paper). Obviously, $L(\lambda)$ is a linearization of $P(\lambda)$ since both have the same elementary divisors. Using (2.10), it can be seen that the signs attached to $1-\delta, 1+\delta$ in the sign characteristic of $L(\lambda)$ are $-1, +1$, respectively.

The standard command `polyeig` in MATLAB applies the QZ algorithm [18] to the first Frobenius companion linearization[1] $C_1(\lambda) = \lambda X + Y$ of the matrix polynomial, and, so, the computed eigenvalues are the exact eigenvalues of $\widetilde{C_1}(\lambda) = \lambda(X + \Delta X) + (Y + \Delta Y)$, where $\|\Delta X\|_2 = O(10^{-16})\|X\|_2$ and $\|\Delta Y\|_2 = O(10^{-16})\|Y\|_2$ in double precision. A fundamental point for our discussion is that *the perturbations ΔX and ΔY do not respect the particular block structure of $C_1(\lambda)$, which implies that $\widetilde{C_1}(\lambda)$ is no longer a Frobenius companion form.* Additionally, the Hermitian nature of the original problem is lost since $C_1(\lambda)$ is not Hermitian. Therefore, other sensible options are to apply to $L(\lambda) = \lambda W_1 + Z_1$ and $D_2(\lambda, P) = \lambda W_2 + Z_2$ algorithms that preserve their Hermitian natures. Such algorithms indeed exist (one can combine for instance the algorithms in [31] and [5]) and, although their backward stability has not been proved, they behave in a backward stable way most of the times. That

[1] In fact, `polyeig` uses the reversal of the first Frobenius companion form of the reversal polynomial.

is, the eigenvalues computed by such Hermitian algorithms are, in practice, the exact eigenvalues of $\lambda(W_i+\Delta W_i)+(Z_i+\Delta Z_i)$, $i=1,2$, in which $\|\Delta W_i\|_2 = O(10^{-16})\|W_i\|_2$ and $\|\Delta Z_i\|_2 = O(10^{-16})\|Z_i\|_2$, where ΔW_i *and* ΔZ_i *are both Hermitian, although do not respect the block structures of* $L(\lambda)$ *and* $D_2(\lambda,P)$, respectively.

Motivated by the discussion above, and taking into account that structure preserving algorithms for Hermitian pencils are not available in MATLAB, the numerical experiments that we present next applied to $P(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ in (3.1) and its linearizations $L(\lambda) = \lambda W_1 + Z_1$, $D_2(\lambda,P) = \lambda W_2 + Z_2$ and $C_1(\lambda) = \lambda X + Y$ have been performed in MATLAB as follows. We have used the command polyeig to compute the eigenvalues of 100 random perturbations (generated with the command randn) $\widetilde{P}(\lambda) = \lambda^2(A_2+\Delta A_2) + \lambda(A_1+\Delta A_1) + (A_0+\Delta A_0)$ of the polynomial in (3.1), and the command eig to compute the eigenvalues of 100 random perturbations of each of its linearizations $\widetilde{L}(\lambda) = \lambda(W_1+\Delta W_1)+(Z_1+\Delta Z_1)$, $\widetilde{D_2}(\lambda,P) = \lambda(W_2+\Delta W_2)+(Z_2+\Delta Z_2)$ and $\widetilde{C_1}(\lambda,P) = \lambda(X+\Delta X)+(Y+\Delta Y)$. The perturbations ΔA_i, $i=0,1,2$, ΔW_j, and ΔZ_j, $j=1,2$ are all Hermitian, but ΔX and ΔY are not. The norms of the perturbations are $\|\Delta A_i\|_2 = O(10^{-7})\|A_i\|_2$, $i=0,1,2$, $\|\Delta W_j\|_2 = O(10^{-7})\|W_j\|_2$, and $\|\Delta Z_j\|_2 = O(10^{-7})\|Z_j\|_2$, $j=1,2$, $\|\Delta X\|_2 = O(10^{-7})\|X\|_2$, and $\|\Delta Y\|_2 = O(10^{-7})\|Y\|_2$. Although this is an experiment on eigenvalue perturbations, the readers may find useful to note that the perturbations of the linearizations mimic the effect of the backward errors of the algorithms discussed in the previous paragraph implemented in simple IEEE precision, that the perturbations on $P(\lambda)$ are performed to ascertain the actual behavior of the eigenvalues of $P(\lambda)$ under Hermitian perturbations, and that the eigenvalues computed by polyeig and eig in double precision are considered as "exact". The results of the experiment are shown in Figure 3.1, where the eigenvalues of all the perturbations described above are plotted in the complex plane. Figure 3.1 shows that the real nature of the close eigenvalues $1-\delta$ and $1+\delta$ is preserved by the perturbations of $P(\lambda)$ and of its sign-characteristic-preserving Hermitian linearization $D_2(\lambda,P)$. In fact, the imaginary parts of all the eigenvalues plotted in these cases are exactly zero. As expected, the perturbations of the Hermitian linearization $L(\lambda)$ do not always preserve the real nature of the close eigenvalues and imaginary parts arise as happens to $C_1(\lambda)$. We would like to emphasize that the eigenvalues of $L(\lambda)$ are all well-conditioned which suggests that the results of the experiment for $L(\lambda)$ only reflect the fact that $L(\lambda)$ does not preserve the sign characteristic of $P(\lambda)$.

The conclusion drawn from the experiment above is that using a Hermitian linearization that preserves the sign characteristic is advantageous for determining the real nature of very close eigenvalues. The explanation of the results in Figure 3.1 relies on the fact that two very close real eigenvalues of a Hermitian matrix polynomial may become nonreal under tiny Hermitian perturbations only if they have opposite signs in the sign characteristic [27]-[30, Theorem 4.7]. As a consequence, the eigenvalues of the perturbations of $P(\lambda)$ and $D_2(\lambda,P)$ remain real, but not always those of the perturbations of $L(\lambda)$, since the eigenvalues 1 ± 10^{-7} have opposite signs in the sign characteristic of $L(\lambda)$. We emphasize that the realness of 1 ± 10^{-7} in $P(\lambda)$ in (3.1) and in $D_2(\lambda,P)$ is surprisingly robust under perturbations, since these eigenvalues remain exactly real even with perturbations of relative size 10^{-3}.

A final remark on the performed numerical tests is that the reader should not get the idea that the linearization $D_2(\lambda,P)$ is always free of difficulties for every matrix polynomial of degree 2. In fact, although it is convenient for studying the realness of the eigenvalues of Hermitian matrix polynomials, if the leading coefficient of the

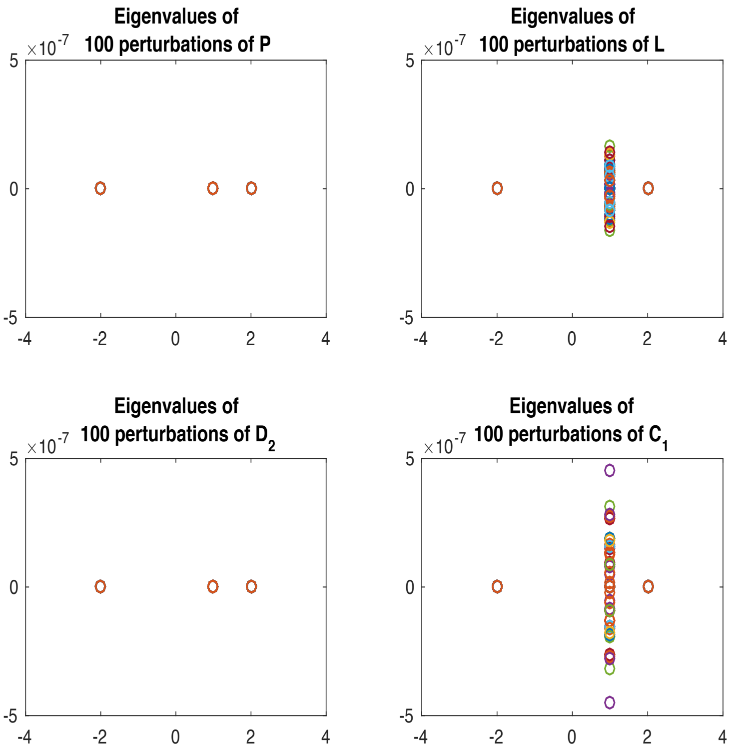

FIG. 3.1. *Eigenvalues of* 100 *random Hermitian perturbations of* $P(\lambda)$ *in* (3.1)*,* 100 *random Hermitian perturbations of its linearization* $L(\lambda)$*,* 100 *random Hermitian perturbations of its linearization* $D_2(\lambda, P)$*, and* 100 *random perturbations of its Frobenius companion form* $C_1(\lambda)$*. In all plots the horizontal axes are the real axis of the complex plane and the vertical axes are the imaginary axis.*

polynomial is very ill-conditioned, then $D_2(\lambda, P)$ is close to not being a linearization of $P(\lambda)$ (see Theorem 4.10) and its eigenvalues may be very sensitive. So, it is of interest to look for other Hermitian linearizations preserving the sign characteristic, as we do in this work. Observe, in this context, that the final backward stable solution of the general unstructured quadratic eigenvalue problem [33] requires the use of *two* different linearizations.

4. Hermitian generalized Fiedler pencils with repetition. In [10], the family of GFPR associated with a matrix polynomial $P(\lambda)$ of degree k as in (2.1) was introduced as an extension of the family of Fiedler pencils with repetition (FPR) presented in [32].[2] A subfamily of block-symmetric GFPR was also identified in that

[2]The key difference between an FPR and a GFPR is that if the coefficients of an FPR are viewed as $k \times k$ block matrices with $n \times n$ blocks, then all the blocks are either 0, $\pm I_n$, or $\pm A_i$, where A_i are the coefficients of $P(\lambda)$. However, in a GFPR arbitrary matrices may also appear among the $n \times n$ blocks.

paper. In Theorem 4.7, we describe the pencils in this subfamily. We will prove in Section 7 that, when $P(\lambda)$ is Hermitian, it contains infinitely many Hermitian strong linearizations of $P(\lambda)$ preserving the sign characteristic of $P(\lambda)$.

4.1. Basic concepts and notation. Next we recall some definitions and notation introduced in [8] and [10] that will be useful in the description of a family of block-symmetric GFPR associated with a matrix polynomial $P(\lambda)$.

We call a finite ordered sequence of integer numbers an index tuple. Each of these integers is called an index. Given an index tuple $\mathbf{t}$ and an integer a, we denote by $a+\mathbf{t}$ the index tuple obtained from $\mathbf{t}$ by adding a to each of its indices. If a and b are two integers, we denote

$$a:b := \begin{cases} a, a+1, \ldots, b, & \text{if } a \leq b, \\ \emptyset, & \text{if } a > b. \end{cases}$$

If c, d are integers, in order to simplify the notation, we denote $(a-c):(b-d)$ by $a-c:b-d$, $a:(b-d)$ by $a:b-d$, and $(a-c):b$ by $a-c:b$.

Definition 4.1. *Given a positive integer m, we call the index tuple $(0:m-1, 0:m-2, \ldots, 0:1, 0)$, formed by the concatenation of m nonempty tuples, the* extended tuple of $(0:m)$, *and denote it by $(0:m)_{ext}$. Also, $(0)_{ext}$ is defined to be the empty tuple.*

Definition 4.2. *Given a nonnegative integer h, we say that an index tuple $\mathbf{t}$ is in* canonical form for h *if $\mathbf{t}$ is empty, when $h = 0, 1$, or if $\mathbf{t}$ is of the form*

$$\left(a_1 : h-2, a_2 : h-4, \ldots, a_{\lfloor \frac{h}{2} \rfloor} : h - 2\left\lfloor \frac{h}{2} \right\rfloor\right) \tag{4.1}$$

when $h \geq 2$, with $0 \leq a_i \leq h - 2i + 1$ (note that if $a_i = h - 2i + 1$, then the string $a_i : h - 2i$ is empty). We say that such a tuple is maximal *if $h = 0, 1$ or $a_i = 0$ for all i. In this case, we denote the tuple by $\mathbf{t}_h$.*

Definition 4.3. *We call the index tuple*

$$(h-1:h, h-3:h-2, \ldots, p+1:p+2, 0:p),$$

where $p = 0$ if h is even and $p = 1$ if h is odd, the admissible tuple associated with the integer $h \geq 0$. *We denote this tuple by $\mathbf{w}_h$.*

Note that the admissible tuple associated with a nonnegative integer h is a permutation of $\{0:h\}$.

Definition 4.4. *Let $h \geq 0$ and $\mathbf{w}_h$ be the admissible tuple associated with h. Define the tuple $\mathbf{c}_h$ as follows:*
- $\mathbf{c}_h = (h-1, h-3, \ldots, 2, 0)$, *if h is odd;*
- $\mathbf{c}_h = (h-1, h-3, \ldots, 1)$, *if $h > 0$ is even;*
- $\mathbf{c}_h = \emptyset$, *if $h = 0$.*

We call the tuple $\mathbf{c}_h$ the symmetric complement of $\mathbf{w}_h$.

The matrix coefficients of the block-symmetric GFPR that we will introduce in Section 4.2 are products of the elementary block-matrices $M_i(B)$ partitioned into $k \times k$ blocks of size $n \times n$ which we define next. For an integer $k \geq 2$ and an $n \times n$ matrix B, let

$$M_0(B) := \left[\begin{array}{c|c} I_{(k-1)n} & 0 \\ \hline 0 & B \end{array}\right], \quad M_{-k}(B) := \left[\begin{array}{c|c} B & 0 \\ \hline 0 & I_{(k-1)n} \end{array}\right],$$

$$
(4.2) \qquad M_i(B) := \left[\begin{array}{c|cc|c} I_{(k-i-1)n} & 0 & 0 & 0 \\ \hline 0 & B & I_n & 0 \\ 0 & I_n & 0 & 0 \\ \hline 0 & 0 & 0 & I_{(i-1)n} \end{array}\right], \quad i = 1 : k-1,
$$

and $M_{-i}(B) = (M_i(-B))^{-1}$, $i = 1 : k-1$. When B is nonsingular, we define in addition $M_k(B) := M_{-k}(B^{-1})$.

Remark 4.5. *It is immediate to check that the commutativity relations*

$$
M_i(B_1)M_j(B_2) = M_j(B_2)M_i(B_1)
$$

hold for any $n \times n$ matrices B_1 and B_2 if $||i| - |j|| > 1$.

Definition 4.6. *Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple with indices contained in $\{-k : k-1\}$ and let $Z := (Z_1, \ldots, Z_r)$ be a list of r arbitrary $n \times n$ matrices. We define*

$$
M_{\mathbf{t}}(Z) := M_{i_1}(Z_1)M_{i_2}(Z_2) \cdots M_{i_r}(Z_r)
$$

and say that Z is a matrix assignment for $\mathbf{t}$. If $\mathbf{t}$ (and therefore Z) is empty, then $M_{\mathbf{t}}(Z) := I_{kn}$. The matrix assignment Z for $\mathbf{t}$ is said to be nonsingular if the matrices assigned to the positions in $\mathbf{t}$ occupied by the 0 and $-k$ indices are nonsingular. If the matrices in Z are Hermitian, then Z is said to be a Hermitian matrix assignment for $\mathbf{t}$.

If $P(\lambda)$ is a matrix polynomial of grade k as in (2.1) and $i \in \{-k : k-1\}$, we will use the following abbreviated notation:

$$
(4.3) \qquad M_i^P := \begin{cases} M_i(-A_i), & \text{if } i \geq 0 \\ M_i(A_{-i}), & \text{if } i < 0 \end{cases}.
$$

Also, if A_k is nonsingular, we define $M_k^P := M_{-k}(A_k^{-1})$. Finally, if $\mathbf{t} = (i_1, ..., i_r)$ is nonempty, we define $M_{\mathbf{t}}^P := M_{i_1}^P M_{i_2}^P \cdots M_{i_r}^P$ and if $\mathbf{t}$ is empty, $M_{\mathbf{t}}^P := I_{kn}$.

4.2. Construction of a family of Hermitian GFPR and related properties. In [10] the GFPR were defined as $kn \times kn$ pencils, where k is the degree of the $n \times n$ matrix polynomial $P(\lambda)$. However, we can define the GFPR in a similar way with k being the grade of $P(\lambda)$ instead of its degree, which is convenient in this paper. Theorem 4.7 was stated in [10] with k being the degree of $P(\lambda)$, but it remains valid if k is the grade and the proof is the same.

Given an ordered list $Z = (Z_1, \ldots, Z_r)$ of arbitrary $n \times n$ matrices, we denote by $rev(Z)$ the following list $rev(Z) = (Z_r, \ldots, Z_1)$.

Theorem 4.7. [10, Theorem 6.11] *Let $P(\lambda)$ be an $n \times n$ matrix polynomial of grade k as in (2.1) and h be an integer such that $0 \leq h < k$. Let $\mathbf{w}_h$ and $\mathbf{w}_{k-h-1}$ be the admissible tuples associated with h and $k-h-1$, respectively, and $\mathbf{c}_h$ and $\mathbf{c}_{k-h-1}$ be the symmetric complements of $\mathbf{w}_h$ and $\mathbf{w}_{k-h-1}$, respectively. Let $\mathbf{v}_h = -k + \mathbf{w}_{k-h-1}$, and let $\mathbf{t}_w$ and $k + \mathbf{t}_v$ be the index tuples in canonical form for h and $k-h-1$, respectively. Let Z_w and Z_v be matrix assignments for $\mathbf{t}_w$ and $\mathbf{t}_v$, respectively. Then, the pencil*

$$
(4.4) \qquad M_{\mathbf{t}_w,\mathbf{t}_v}(Z_w, Z_v)\,(\lambda M_{\mathbf{v}_h}^P - M_{\mathbf{w}_h}^P)M_{-k+\mathbf{c}_{k-h-1},\mathbf{c}_h}^P M_{rev(\mathbf{t}_w),rev(\mathbf{t}_v)}(rev(Z_w), rev(Z_v))
$$

is a block-symmetric GFPR and will be denoted by $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$.

If $L(\lambda) := \lambda L_1 - L_0$ is a pencil as in (4.4) and L_1 and L_0 are viewed as $k \times k$ block matrices whose blocks are of size $n \times n$, then each of these blocks is either $0, I_n, \pm A_i$, or one of the matrices in the matrix assignments Z_w, Z_v [10, Theorem 5.3]. We stress the fact that the block symmetry mentioned in Theorem 4.7 considers this partition and means that $L_1^{\mathcal{B}} = L_1$ and $L_0^{\mathcal{B}} = L_0$.

Note that, if Z_w and Z_v are chosen so that $M_{\mathbf{t}_w}(Z_w) = M^P_{\mathbf{t}_w}$ and $M_{\mathbf{t}_v}(Z_v) = M^P_{\mathbf{t}_v}$, then $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ is a block-symmetric FPR, which we denote by $L_P(h, \mathbf{t}_w, \mathbf{t}_v)$.

REMARK 4.8. *Based on results from [32], it was observed in [10, Remark 6.2] that the pencils in the standard basis of the space of block-symmetric pencils $\mathbb{DL}(P)$ [19, Section 3.3] are FPR. Let us denote by $D_m(\lambda, P)$, for $m = 1 : k$, the mth pencil in this basis. Then*

$$D_m(\lambda, P) = L_P(k - m, \mathbf{t}_{k-m}, -k + \mathbf{t}_{m-1}), \tag{4.5}$$

where $\mathbf{t}_{k-m}$ and $\mathbf{t}_{m-1}$ are the maximal index tuples in canonical form for $k - m$ and $m - 1$, respectively (see [32, Corollaries 1 and 2] and [7, Lemma 5.7]). Moreover, a direct computation shows that also

$$D_m(\lambda, P) = \lambda M^P_{(0:k-m)_{ext}} M^P_{-k+(0:m)_{ext}} - M^P_{(0:k-m+1)_{ext}} M^P_{-k+(0:m-1)_{ext}}. \tag{4.6}$$

The following result establishes when a block-symmetric GFPR as in (4.4) is a strong linearization of a regular matrix polynomial $P(\lambda)$ of grade k.

THEOREM 4.9. *[10, Theorem 5.5] Let $P(\lambda)$ be an $n \times n$ regular matrix polynomial of grade k as in (2.1). The pencil $L_P(h, t_w, t_v, Z_w, Z_v)$ introduced in Theorem 4.7 is a strong linearization of $P(\lambda)$ if and only if the following three conditions hold simultaneously:*

(i) Z_w and Z_v are nonsingular matrix assignments for t_w and t_v, respectively,
(ii) A_0 is nonsingular if h is odd, and
(iii) A_k is nonsingular if $k - h$ is even.

From Theorem 4.9 one can easily obtain necessary and sufficient conditions for the pencils $D_m(\lambda, P)$ in the standard basis of $\mathbb{DL}(P)$ to be strong linearizations of $P(\lambda)$. These conditions can also be immediately obtained from the eigenvalue exclusion theorem [21, Theorem 6.7] and are stated in Theorem 4.10. We omit the trivial proof.

THEOREM 4.10. *Let $P(\lambda)$ be a regular matrix polynomial of grade k as in* (2.1) *and let $D_m(\lambda, P)$ be the pencils in the standard basis of $\mathbb{DL}(P)$ for $m = 1 : k$. Then:*

(a) $D_1(\lambda, P)$ is a strong linearization of $P(\lambda)$ if and only if A_0 is nonsingular.
(b) For $1 < m < k$, $D_m(\lambda, P)$ is a strong linearization of $P(\lambda)$ if and only if A_0 and A_k are nonsingular.
(c) $D_k(\lambda, P)$ is a strong linearization of $P(\lambda)$ if and only if A_k is nonsingular.

Next we associate to each block-symmetric GFPR, in particular to each pencil in the standard basis of $DL(P)$, another block-symmetric GFPR with empty tuples t_w and t_v. These pencils play an important role in this paper and its simple block structure is described in Section 8.

DEFINITION 4.11. *Let $P(\lambda)$ be a matrix polynomial of grade k as in (2.1). We call the pencil $L_P(h, \emptyset, \emptyset) = (\lambda M^P_{\mathbf{v}_h} - M^P_{\mathbf{w}_h}) M^P_{-k+\mathbf{c}_{k-h-1}, \mathbf{c}_h}$ the simple FPR associated with h. Also, given the block-symmetric GFPR $L(\lambda) = L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ introduced in Theorem 4.7, we say that $L_P(h, \emptyset, \emptyset)$ is the simple FPR associated with $L(\lambda)$.*

Theorem 4.12 relates properties of a block-symmetric GFPR and its associated simple FPR. The proof follows easily from Theorem 4.9 and is omitted.

THEOREM 4.12. *Let $P(\lambda)$ be a regular matrix polynomial of grade k as in (2.1), let $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ be the block-symmetric GFPR introduced in Theorem 4.7, and let $L_P(h, \emptyset, \emptyset)$ be its associated simple FPR. Then the following statements hold.*

(a) $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ is a strong linearization of $P(\lambda)$ if and only if $L_P(h, \emptyset, \emptyset)$ is a strong linearization of $P(\lambda)$ and Z_w, Z_v are nonsingular matrix assignments for $\mathbf{t}_w, \mathbf{t}_v$, respectively.

(b) $L_P(h, \emptyset, \emptyset)$ is a strong linearization of $P(\lambda)$ if and only if A_0 is nonsingular when h is odd and A_k is nonsingular when $k - h$ is even.

We illustrate the concepts and results introduced above in Example 4.13.

EXAMPLE 4.13. *Let $P(\lambda)$ be a matrix polynomial of grade $k = 4$ as in (2.1). Then, the simple FPR associated with $D_1(\lambda, P) = L_P(3, t_3, \emptyset)$ is given by $L_P(3, \emptyset, \emptyset) = \left(\lambda M^P_{-4} - M^P_{2:3,0:1}\right) M^P_{2,0}$, which can be expressed explicitly as*

$$L_P(3, \emptyset, \emptyset) = \lambda \begin{bmatrix} A_4 & 0 & 0 & 0 \\ 0 & -A_2 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & -A_0 \end{bmatrix} - \begin{bmatrix} -A_3 & -A_2 & I_n & 0 \\ -A_2 & -A_1 & 0 & -A_0 \\ I_n & 0 & 0 & 0 \\ 0 & -A_0 & 0 & 0 \end{bmatrix}.$$

According to Theorem 4.10(a), $D_1(\lambda, P)$ is a strong linearization of $P(\lambda)$ if and only if A_0 is nonsingular. Then, Theorem 4.12(a) immediately guarantees that $L_P(3, \emptyset, \emptyset)$ is also a strong linearization of $P(\lambda)$ if A_0 is nonsingular. Theorem 4.12(b) guarantees more: $L_P(3, \emptyset, \emptyset)$ is a strong linearization of $P(\lambda)$ if and only if A_0 is nonsingular.

Since a pencil $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ as in Theorem 4.7 is block-symmetric for any $P(\lambda)$ and the blocks of its coefficients are either $0, I_n, \pm A_i$, or one of the matrices in Z_w and Z_v, we obtain the following result.

COROLLARY 4.14. *Let $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ be a block-symmetric GFPR as in Theorem 4.7. If $P(\lambda)$ is Hermitian and the matrix assignments Z_w, Z_v are Hermitian, then $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ is Hermitian and is referred to as a HGFPR.*

Note, in particular, that if $P(\lambda)$ is Hermitian, then $L_P(h, \emptyset, \emptyset)$ is a Hermitian FPR.

The following theorem will be very useful later.

THEOREM 4.15. *Let $P(\lambda)$ be an $n \times n$ Hermitian matrix polynomial of grade k and let $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ be a HGFPR associated with $P(\lambda)$. If the Hermitian matrix assignments Z_w for $\mathbf{t}_w$ and Z_v for $\mathbf{t}_v$ are nonsingular, then $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ is *congruent to its associated simple FPR $L_P(h, \emptyset, \emptyset)$. More precisely,*

$$L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v) = Q\, L_P(h, \emptyset, \emptyset)\, Q^*,$$

where $Q = M_{\mathbf{t}_w, \mathbf{t}_v}(Z_w, Z_v)$ is nonsingular.

Proof. From the definition of the elementary matrices $M_i(B)$ (see (4.2) and the expressions for those matrices above and below (4.2)), we see that $M_i(B)^* = M_i(B^*) = M_i(B)$ if $B = B^*$. In addition, note that $M_{\mathbf{t}_w}(Z_w) M_{\mathbf{t}_v}(Z_v) = M_{\mathbf{t}_v}(Z_v) M_{\mathbf{t}_w}(Z_w)$, because $t_w \subseteq \{0 : h-2\}$, $t_v \subseteq \{-k : -(h+3)\}$, and Remark 4.5 holds. Therefore,

$$\begin{aligned} (M_{\mathbf{t}_w, \mathbf{t}_v}(Z_w, Z_v))^* &= (M_{\mathbf{t}_w}(Z_w) M_{\mathbf{t}_v}(Z_v))^* = (M_{\mathbf{t}_v}(Z_v) M_{\mathbf{t}_w}(Z_w))^* \\ &= (M_{\mathbf{t}_w}(Z_w))^* \, (M_{\mathbf{t}_v}(Z_v))^* = M_{rev(\mathbf{t}_w)}(rev(Z_w))\, M_{rev(\mathbf{t}_v)}(rev(Z_v)) \\ &= M_{rev(\mathbf{t}_w), rev(\mathbf{t}_v)}(rev(Z_w), rev(Z_v)). \end{aligned}$$

The rest of the proof follows from the form of $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ described in (4.4) and the nonsingularity of the matrix assignments Z_w and Z_v. $\square$

5. Characterization of the strong linearizations that preserve the sign characteristic. In this section we characterize in Theorem 5.3 all the Hermitian strong linearizations of a Hermitian matrix polynomial $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$ with A_k nonsingular that preserve its sign characteristic. Based on this general characterization, we identify in Sections 6 and 7 many Hermitian strong linearizations of $P(\lambda)$ which are easily constructible from the coefficients of $P(\lambda)$ and that preserve its sign characteristic. Clearly, these linearizations are privileged for working numerically with $P(\lambda)$, since they preserve all its relevant spectral properties.

The next lemma characterizes Hermitian pencils with the same elementary divisors and sign characteristic. Its proof is omitted since it follows from [20, Thms. 6.1-12.1] by noting that the concept of sign characteristic of a matrix pencil presented here coincides essentially with the one in [20]. In fact, the only difference is that the signs in the sign characteristic of a Hermitian matrix pencil associated with a Jordan block of even size given by our definition and by the one in [20] are opposite.

LEMMA 5.1. *Let $L(\lambda) = \lambda L_1 - L_0$ and $\widehat{L}(\lambda) = \lambda \widehat{L}_1 - \widehat{L}_0$ be two complex Hermitian pencils such that L_1 and $\widehat{L}_1$ are nonsingular. Then, $L(\lambda)$ and $\widehat{L}(\lambda)$ have the same elementary divisors and the same sign characteristic if and only if $L(\lambda)$ and $\widehat{L}(\lambda)$ are *congruent.*

The block-symmetric kth pencil in the standard basis of $\mathbb{DL}(P)$ is fundamental in this section. Recall that, by (4.6) in Remark 4.8,

$$D_k(\lambda, P) = \lambda M^P_{-k+(0:k)_{ext}} - M^P_0 M^P_{-k+(0:k-1)_{ext}}. \tag{5.1}$$

In [2, Lemma 2.8], it was proven that, if $P(\lambda)$ is a Hermitian matrix polynomial with nonsingular leading coefficient whose eigenvalues are all real and of definite type (and, so, they are semisimple), then $D_k(\lambda, P)$ is a Hermitian linearization that preserves the sign characteristic of $P(\lambda)$. In Lemma 5.2, we show that $D_k(\lambda, P)$ also preserves the sign characteristic of $P(\lambda)$ when there is no restriction on the eigenvalues of $P(\lambda)$.

LEMMA 5.2. *Let $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$ be an $n \times n$ Hermitian matrix polynomial with A_k nonsingular and let $D_k(\lambda, P)$ be the kth pencil of the standard basis of $\mathbb{DL}(P)$. Then $D_k(\lambda, P)$ is a Hermitian strong linearization of $P(\lambda)$ that preserves the sign characteristic of $P(\lambda)$.*

Proof. It is well-known that $D_k(\lambda, P)$ is a Hermitian strong linearization of $P(\lambda)$ [19, 21, 22]. This fact also follows from Remark 4.8, Corollary 4.14, and Theorem 4.10. Next, we prove that $D_k(\lambda, P)$ has the same sign characteristic as $P(\lambda)$. An easy computation gives

$$M^P_{0:k} = \begin{bmatrix} -A_{k-1}A_k^{-1} & I_n & 0 & \cdots & 0 \\ -A_{k-2}A_k^{-1} & 0 & I_n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_1 A_k^{-1} & 0 & 0 & \cdots & I_n \\ -A_0 A_k^{-1} & 0 & 0 & \cdots & 0 \end{bmatrix} = R_k C_P^* R_k,$$

where R_k is the $k \times k$ block-sip matrix with $n \times n$ blocks (2.7) and C_P was defined in (2.4). Let $\lambda L_1 - L_0 := D_k(\lambda, P)$. Taking into account (5.1) and the well-known fact [19, Theorem 3.5] that $L_1 = R_k B_P R_k$, where B_P was defined in (2.5), it follows that

$$M^P_{-k+(0:k)_{ext}} = R_k B_P R_k.$$

Since $M_0^P M^P_{-k+(0:k-1)_{ext}} = M^P_{0:k} M^P_{-k+(0:k)_{ext}}$, we get[3]

$$D_k(\lambda, P) = R_k(\lambda B_P - C_P^* B_P) R_k. \tag{5.2}$$

Since $R_k^* = R_k$ and $D_k(\lambda, P)$ is Hermitian, we see from (5.2) that the pencil $H(\lambda) = \lambda B_P - C_P^* B_P$ is also Hermitian and from Lemma 5.1 that the sign characteristic of $D_k(\lambda, P)$ is equal to the sign characteristic of $\lambda B_P - C_P^* B_P$. Since $C_P^* B_P$ and B_P are Hermitian matrices, $C_H = B_P^{-1} C_P^* B_P = B_P^{-1}(C_P^* B_P)^* = C_P$. On the other hand, $B_H = B_P$. Thus, from the definition of sign characteristic, $H(\lambda)$ and $P(\lambda)$ have the same sign characteristic. □

Theorem 5.3 characterizes all the Hermitian strong linearizations that preserve the sign characteristic of a Hermitian matrix polynomial $P(\lambda)$ with nonsingular leading coefficient: they are precisely those pencils that are *congruent to $D_k(\lambda, P)$. The proof follows easily from Lemmas 5.1 and 5.2 and is omitted.

Theorem 5.3. *Let $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$ be an $n \times n$ Hermitian matrix polynomial with A_k nonsingular and let $D_k(\lambda, P)$ be the kth pencil of the standard basis of $\mathbb{DL}(P)$. Then, an $nk \times nk$ pencil $L(\lambda)$ is a Hermitian strong linearization of $P(\lambda)$ that preserves the sign characteristic of $P(\lambda)$ if and only if $L(\lambda)$ is *congruent to $D_k(\lambda, P)$.*

6. Hermitian block-tridiagonal strong linearizations that preserve the sign characteristic. We consider in this section two very well known block-symmetric strong linearizations of a matrix polynomial $P(\lambda)$ as in (2.1) which are Hermitian when $P(\lambda)$ is, and, in this case, we apply Theorem 5.3 to prove that they preserve the sign characteristic of $P(\lambda)$. These two linearizations are essentially the same. The first one was introduced in [3, Theorem 3.1] and the second one is a variation of the first when $P(\lambda)$ has odd grade, which is simpler and has been used in [23, pp. 884-887], [24, pp. 81-84], and [25, pp. 4646-4647] to construct structure preserving linearizations for several important classes of structured matrix polynomials. These linearizations are, probably, the simplest block-symmetric strong linearizations associated with $P(\lambda)$ of odd grade, since they are block-tridiagonal, are very easily constructible from the coefficients of $P(\lambda)$ without doing any operations, and each coefficient of $P(\lambda)$ appears exactly once in the linearizations. The reader is invited to check this simplicity in the references cited above. In addition, they allow us to recover very easily the eigenvectors of $P(\lambda)$ from these linearizations [9, Corollary 3.6]. Another interesting property of the block-tridiagonal linearizations introduced in [3] and [23, 24, 25] is that, if the grade of $P(\lambda)$ is odd, then they are *companion forms* [13, Definition 5.1], so, in particular, they are strong linearizations for any $P(\lambda)$, independently of the properties of its coefficients. We emphasize that these linearizations are not included in the family of block-symmetric GFPR considered in Theorem 4.7, neither are FPR. They belong to the family of generalized Fiedler pencils introduced in [9, Definition 2.1].

Given a general matrix polynomial $P(\lambda)$ as in (2.1), we give in (6.1) the block-tridiagonal pencil introduced in [3, Theorem 3.1] in terms of the elementary matrices in (4.3). Note that if k is even, (6.1) requires A_k to be nonsingular and that A_k^{-1} appears as a block in the zero-degree coefficient of $L(\lambda)$.

$$L(\lambda) := \begin{cases} \lambda M^P_{-1,-3,\ldots,-k+2,-k} - M^P_{0,2,\ldots,k-1}, & \text{if } k \text{ is odd,} \\ \lambda M^P_{-1,-3,\ldots,-k+1} - M^P_{0,2,\ldots,k}, & \text{if } k \text{ is even.} \end{cases} \tag{6.1}$$

[3] Equation (5.2) can be obtained also from the explicit block-expression of the coefficients of $D_k(\lambda, P)$ in [19, Theorem 3.5] and the expressions of C_P and B_P.

Equation (6.2) gives the expression of the block-tridiagonal pencil introduced in [23, pp. 884-887], which is valid only if k is odd:

$$\widetilde{L}(\lambda) := R_k S(\lambda M^P_{-1,-3,\ldots,-k+2,-k} - M^P_{0,2,\ldots,k-1}) S R_k, \tag{6.2}$$

where R_k is the $k \times k$ block-sip matrix (2.7) and S is a $k \times k$ block-diagonal matrix whose (i,i) block-entry is given by

$$S(i,i) = \begin{cases} -I_n, & \text{if } i = 0,1 \bmod 4 \\ I_n, & \text{otherwise.} \end{cases}$$

As an illustration, we write the pencils $L(\lambda)$ and $\widetilde{L}(\lambda)$ explicitly for degree 5.

Example 6.1. *Let $P(\lambda) = \sum_{i=0}^{5} A_i \lambda^i$ be a Hermitian matrix polynomial. Then,*

$$L(\lambda) = \begin{bmatrix} \lambda A_5 + A_4 & -I & 0 & 0 & 0 \\ -I & 0 & \lambda I & 0 & 0 \\ 0 & \lambda I & \lambda A_3 + A_2 & -I & 0 \\ 0 & 0 & -I & 0 & \lambda I \\ 0 & 0 & 0 & \lambda I & \lambda A_1 + A_0 \end{bmatrix},$$

$$\widetilde{L}(\lambda) = \begin{bmatrix} \lambda A_1 + A_0 & \lambda I & 0 & 0 & 0 \\ \lambda I & 0 & I & 0 & 0 \\ 0 & I & \lambda A_3 + A_2 & \lambda I & 0 \\ 0 & 0 & \lambda I & 0 & I \\ 0 & 0 & 0 & I & \lambda A_5 + A_4 \end{bmatrix}.$$

Next, we state and prove the main result in this section.

Theorem 6.2. *Let $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$ be an $n \times n$ Hermitian matrix polynomial with A_k nonsingular. Then, the pencil $L(\lambda)$, defined in (6.1), and the pencil $\widetilde{L}(\lambda)$, defined in (6.2) if k odd, are Hermitian strong linearizations of $P(\lambda)$ that preserve the sign characteristic of $P(\lambda)$.*

Proof. Let us start by proving that $L(\lambda)$ is a Hermitian strong linearization of $P(\lambda)$ that preserves its sign characteristic. It is well known that $L(\lambda)$ is a strong linearization of $P(\lambda)$, since it is strictly equivalent to a standard Fiedler pencil, which is a strong linearization of $P(\lambda)$ [3, 9]. It is also well known that $L(\lambda)$ is Hermitian, when $P(\lambda)$ is (it follows immediately from (6.1), (4.3), and Remark 4.5). Next, define

$$W := \begin{cases} M^P_{-2,-4\ldots,-k+1}, & \text{if } k \text{ is odd,} \\ M^P_{-2,-4,\ldots,-k}, & \text{if } k \text{ is even,} \end{cases}$$

which is a nonsingular Hermitian matrix. In addition, by Definition 4.11, the simple FPR $L_P(0,\emptyset,\emptyset)$ is given by

$$L_P(0,\emptyset,\emptyset) = \begin{cases} (\lambda M^P_{-2:-1,\ldots,-k+1:-k+2,-k} - M^P_0) M^P_{-2,-4,\ldots,-k+1}, & \text{if } k \text{ is odd,} \\ (\lambda M^P_{-2:-1,\ldots,-k:-k+1} - M^P_0) M^P_{-2,-4,\ldots,-k}, & \text{if } k \text{ is even,} \end{cases}$$

and a simple computation shows that

$$W L(\lambda) W^* = L_P(0,\emptyset,\emptyset). \tag{6.3}$$

On the other hand, $D_k(\lambda,P)$ is a HGFPR by Corollary 4.14 and $L_P(0,\emptyset,\emptyset)$ is its associated simple FPR. Thus, $D_k(\lambda,P)$ and $L_P(0,\emptyset,\emptyset)$ are *congruent by Theorem

4.15, as condition (i) in Theorem 4.9 holds for $D_k(\lambda, P)$. This last statement together with (6.3) imply that $L(\lambda)$ is *congruent to $D_k(\lambda, P)$. Therefore, from Theorem 5.3, we get that $L(\lambda)$ is a Hermitian strong linearization of $P(\lambda)$ that preserves its sign characteristic.

Next, consider $\widetilde{L}(\lambda)$ in (6.2). Since $(SR_k)^* = R_k^* S^* = R_k S$, $\widetilde{L}(\lambda)$ is *congruent to $L(\lambda)$ and, so, is *congruent to $D_k(\lambda, P)$. Theorem 5.3 guarantees again that $\widetilde{L}(\lambda)$ is a Hermitian strong linearization of $P(\lambda)$ that preserves its sign characteristic. □

7. HGFPR strong linearizations that preserve the sign characteristic. The main result of this section is Theorem 7.1, which gives a class of HGFPR (recall Corollary 4.14) which are strong linearizations of a Hermitian matrix polynomial $P(\lambda)$ and preserve its sign characteristic. The proof of Theorem 7.1 follows again from Theorem 5.3. After Theorem 7.1, we present Corollary 7.2, which proves that other pencils different from $D_k(\lambda, P)$ in the standard basis of $\mathbb{DL}(P)$ also preserve the sign characteristic of $P(\lambda)$.

THEOREM 7.1. *Let $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$ be an $n \times n$ Hermitian matrix polynomial with A_k nonsingular, let h be an even integer such that $0 \le h < k$, and let $L(\lambda) = L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ be an HGFPR for $P(\lambda)$ where Z_w and Z_v are nonsingular Hermitian matrix assignments for $\mathbf{t}_w$ and $\mathbf{t}_v$, respectively. Then, $L(\lambda)$ is a Hermitian strong linearization of $P(\lambda)$ that preserves the sign characteristic of $P(\lambda)$.*

Proof. We will prove that $L(\lambda)$ is *congruent to $D_k(\lambda, P)$, which combined with Theorem 5.3 proves the result. By Theorem 4.15, $L(\lambda)$ is *congruent to $L_P(h, \emptyset, \emptyset)$. By Theorem 4.15, (4.5), and Theorem 4.9(i), we get also that $D_k(\lambda, P)$ is *congruent to $L_P(0, \emptyset, \emptyset)$. Therefore, by Lemma 5.2, it is enough to show that $L_P(h, \emptyset, \emptyset)$ and $L_P(0, \emptyset, \emptyset)$ are *congruent. If $h = 0$, it is obvious. So, in the rest of the proof, we suppose that $h \neq 0$ and h is even. If k is odd, we have

$$L(0, \emptyset, \emptyset) = (\lambda M^P_{-2:-1,\ldots,-k+1:-k+2,-k} - M^P_0) M^P_{-2,-4,\ldots,-k+1},$$

$$\begin{aligned} &L_P(h, \emptyset, \emptyset) \\ &= (\lambda M^P_{-h-2:-h-1,\ldots,-k+1:-k+2,-k} - M^P_{h-1:h,\ldots 1:2,0}) M^P_{1,3,\ldots,h-1} M^P_{-h-2,-h-4,\ldots,-k+1}. \end{aligned}$$

If k is even, we have

$$\begin{aligned} L(0, \emptyset, \emptyset) &= (\lambda M^P_{-2:-1,\ldots,-k:-k+1} - M^P_0) M^P_{-2,-4,\ldots,-k}, \\ L_P(h, \emptyset, \emptyset) &= (\lambda M^P_{-h-2:-h-1,\ldots,-k:-k+1} - M^P_{h-1:h,\ldots 1:2,0}) M^P_{1,3,\ldots,h-1} M^P_{-h-2,-h-4,\ldots,-k}. \end{aligned}$$

In both cases, it can be easily verified via a direct computation that

$$QL(0, \emptyset, \emptyset)Q^* = L_P(h, \emptyset, \emptyset), \tag{7.1}$$

with $Q = M^P_{h-1:h,\ldots,3:4,1:2} = (M^P_{-2:-1,-4:-3\ldots,-h:-h+1})^{-1}$ a nonsingular matrix. □

Next, we obtain Corollary 7.2 as a consequence of Theorem 7.1.

COROLLARY 7.2. *Let $P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i$ be a Hermitian matrix polynomial and $m \in \{1 : k\}$. Suppose that A_k is nonsingular, and A_0 is nonsingular if $m \neq k$. Let $D_m(\lambda, P)$ be the mth pencil in the standard basis of $\mathbb{DL}(P)$. If $k - m$ is even, then $D_m(\lambda, P)$ is a Hermitian strong linearization of $P(\lambda)$ with the same sign characteristic as $P(\lambda)$.*

Proof. The case $m = k$ follows from Lemma 5.2. Let us focus then on the case $m = 1 : k - 1$. From (4.5) and Corollary 4.14 we get that $D_m(\lambda, P)$ is a Hermitian pencil. In addition, from the nonsingularity of A_0 and A_k, the Hermitian matrix assignments for $D_m(\lambda, P) = L_P(k-m, \mathbf{t}_{k-m}, -k+\mathbf{t}_{m-1})$ are nonsingular. The result then follows from Theorem 7.1. $\square$

Note that, if $k = 2$, Theorem 7.1 only provides one sign-characteristic-preserving linearization of $P(\lambda)$, namely, $D_2(\lambda, P)$. However, if $k \geq 3$, Theorem 7.1 provides an infinite set containing such sign-characteristic-preserving strong linearizations of $P(\lambda)$ as we show in the following examples.

EXAMPLE 7.3. *Let $P(\lambda) = \sum_{i=0}^{3} A_i \lambda^i$ be a Hermitian matrix polynomial of degree 3 with A_3 nonsingular. By Theorem 7.1, all the HGFPR of the form*

$$L_P(2, (0), \emptyset, (Z_1), \emptyset) = \lambda \begin{bmatrix} A_3 & 0 & 0 \\ 0 & -A_1 & Z_1 \\ 0 & Z_1 & 0 \end{bmatrix} - \begin{bmatrix} -A_2 & -A_1 & Z_1 \\ -A_1 & -A_0 & 0 \\ Z_1 & 0 & 0 \end{bmatrix},$$

where Z_1 is a nonsingular Hermitian matrix, are Hermitian strong linearizations of $P(\lambda)$ that preserve the sign characteristic of $P(\lambda)$.

EXAMPLE 7.4. *Let $P(\lambda)$ be a Hermitian matrix polynomial of degree 5 as in (2.1), with A_5 nonsingular. A set of HGFPR which are strong linearizations and preserve the sign characteristic of $P(\lambda)$, as follows from Theorem 7.1, is given by the pencils of the form $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$, where Z_w and Z_v are nonsingular Hermitian matrix assignments for $\mathbf{t}_w$ and $\mathbf{t}_v$, respectively, and h, $\mathbf{t}_w$, and $\mathbf{t}_v$ are one of the following:*

- *$h = 0$, $\mathbf{t}_w = \emptyset$ and*

$$\begin{array}{lll} \mathbf{t}_v = \emptyset; & \mathbf{t}_v = (-5); & \mathbf{t}_v = (-3, -5); \\ \mathbf{t}_v = (-4, -3, -5); & \mathbf{t}_v = (-5 : -3, -5); & \mathbf{t}_v = (-3); \\ \mathbf{t}_v = (-4 : -3); & \mathbf{t}_v = (-5 : -3). & \end{array}$$

- *$h = 2$,*

$$\begin{array}{ll} \mathbf{t}_w = \mathbf{t}_v = \emptyset; & \mathbf{t}_w = (0),\ \mathbf{t}_v = \emptyset; \\ \mathbf{t}_w = \emptyset,\ \mathbf{t}_v = (-5); & \mathbf{t}_w = (0),\ \mathbf{t}_v = (-5). \end{array}$$

- *$h = 4$, $\mathbf{t}_v = \emptyset$ and*

$$\begin{array}{lll} \mathbf{t}_w = \emptyset; & \mathbf{t}_w = (0); & \mathbf{t}_w = (2, 0); \\ \mathbf{t}_w = (1 : 2, 0); & \mathbf{t}_w = (0 : 2, 0); & \mathbf{t}_w = (2); \\ \mathbf{t}_w = (1 : 2); & \mathbf{t}_w = (0 : 2). & \end{array}$$

Observe that, for each combination of h, $\mathbf{t}_w$, and $\mathbf{t}_v$, if either $\mathbf{t}_w$ or $\mathbf{t}_v$ is not empty, we get an infinite family of sign-preserving strong linearizations.

REMARK 7.5. To end this section, we consider whether Theorem 7.1 can be extended to odd values of h, and Corollary 7.2 can be extended to odd values of $k - m$. We show via Example 7.6 that, in general, such extension is not possible. In order to understand Example 7.6, we need some preliminary arguments. If $P(\lambda)$ is a Hermitian matrix polynomial of degree k as in (2.1), with A_0 and A_k nonsingular, and h is odd, an argument similar to the one used in the proof of Theorem 7.1 for getting (7.1) shows that $L_P(h, \emptyset, \emptyset)$ is *congruent to $L_P(1, \emptyset, \emptyset)$. Observe that the nonsingularity of A_0 and A_k ensures that $L_P(h, \emptyset, \emptyset)$ is a strong linearization of $P(\lambda)$, by Theorem 4.12. Since, by Theorem 4.15, any HGFPR $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ with nonsingular matrix assignments Z_w and Z_v is *congruent to $L_P(h, \emptyset, \emptyset)$, we get that any HGFPR $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ with h odd and which is a strong linearization of

$P(\lambda)$ is *congruent to $L_P(1,\emptyset,\emptyset)$. So, from Lemma 5.1, it follows that all HGFPR $L_P(h,\mathbf{t}_w,\mathbf{t}_v,Z_w,Z_v)$ with h odd which are strong linearizations of $P(\lambda)$ preserve the sign characteristic of $P(\lambda)$ if $L_P(1,\emptyset,\emptyset)$ does, and none of them preserves the sign characteristic of $P(\lambda)$ if $L_P(1,\emptyset,\emptyset)$ does not. In addition, Theorem 7.1 implies that $L_P(0,\emptyset,\emptyset)$ is an HGFPR-strong-linearization that preserves the sign characteristic of $P(\lambda)$. Therefore, from the discussion above and Lemma 5.1, we get that an HGFPR $L_P(h,\mathbf{t}_w,\mathbf{t}_v,Z_w,Z_v)$ with h odd is a strong linearization of $P(\lambda)$ preserving the sign characteristic of $P(\lambda)$ if and only if $L_P(1,\emptyset,\emptyset)$ and $L_P(0,\emptyset,\emptyset)$ are *congruent. In Example 7.6, we present a matrix polynomial $P(\lambda)$ as above for which these two pencils are not *congruent, implying that no HGFPR linearization $L_P(h,\mathbf{t}_w,\mathbf{t}_v,Z_w,Z_v)$ of $P(\lambda)$, with h odd, preserves the sign characteristic of this particular $P(\lambda)$.

It can be seen that $L_P(1,\emptyset,\emptyset)$ and $L_P(0,\emptyset,\emptyset)$ are *congruent for some other Hermitian matrix polynomials $P(\lambda)$ with nonsingular leading coefficient, implying in this case that all HGFPR linearizations of $P(\lambda)$ preserve its sign characteristic.

Example 7.6. *Consider the Hermitian matrix polynomial of degree* 3 *with nonsingular leading coefficient:*

$$P(\lambda) = \begin{bmatrix} \lambda^3-4 & 0 \\ 0 & \lambda^3+1 \end{bmatrix}. \tag{7.2}$$

Let $L_P(1,\emptyset,\emptyset) := \lambda L_1 - L_0$ *and* $L_P(0,\emptyset,\emptyset) := \lambda L_1' - L_0'$. *Then, from Definition 4.11, we have* $L_0 = M^P_{0,1,0,-3}$ *and* $L_0' = M^P_{0,-2}$, *which for* $P(\lambda)$ *in* (7.2) *yields*

$$L_0 = \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right] \quad \text{and} \quad L_0' = \left[\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right].$$

It can be easily verified that the eigenvalues of L_0 *are* $-4,-1,1,1,1,4$, *while the eigenvalues of* L_0' *are* $-1,-1,-1,1,1,4$. *Thus,* L_0 *and* L_0' *have different inertias and, therefore, by the Sylvester's Law of Inertia, they are not *congruent. Thus,* $L_P(1,\emptyset,\emptyset)$ *and* $L_P(0,\emptyset,\emptyset)$ *are not *congruent, implying that no HGFPR* $L_P(h,\mathbf{t}_w,\mathbf{t}_v,Z_w,Z_v)$ *with* h *odd which is a strong linearization of* $P(\lambda)$ *preserves its sign characteristic, although all these pencils have the same sign characteristic.*

8. Block structure of simple FPR. In this section, we describe the block structure of the simple FPRs, $L_P(h,\emptyset,\emptyset)$, associated with a matrix polynomial $P(\lambda)$ as in (2.1) that were introduced in Definition 4.11 and have been used very often in this paper. This description complements the representation of simple FPRs in terms of products of elementary matrices and allows us to construct easily new Hermitian strong linearizations that preserve the sign characteristic of $P(\lambda)$ when $P(\lambda)$ is Hermitian and its leading coefficient is nonsingular. In this case, recall that any $L_P(h,\emptyset,\emptyset)$ with h even is such a linearization according to Theorem 7.1.

The description of $L_P(h,\emptyset,\emptyset)$ in the main result of this section (i.e., Theorem 8.1) uses some matrices associated with $P(\lambda)=\sum_{i=0}^{k} A_i\lambda^i$ which are introduced below:

$$D_i := \begin{bmatrix} -A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad E_i := \begin{bmatrix} -A_i & I_n \\ 0 & 0 \end{bmatrix}, \quad F_i := \begin{bmatrix} -A_i & I_n \\ I_n & 0 \end{bmatrix}, \quad 0 \le i < k,$$

$$T_1 := \begin{bmatrix} -A_1 & 0 & -A_0 \\ 0 & 0 & 0 \\ -A_0 & 0 & 0 \end{bmatrix}, \quad T_h := \begin{bmatrix} -A_h & -A_{h-1} & I_n \\ -A_{h-1} & -A_{h-2} & 0 \\ I_n & 0 & 0 \end{bmatrix}, \quad 2 \le h < k,$$

where all blocks have size $n \times n$. Based on these matrices, we define in addition

$$C_{0,P} := -A_0, \quad C_{1,P} := \begin{bmatrix} -A_1 & -A_0 \\ -A_0 & 0 \end{bmatrix}, \quad C_{2,P} := T_2, \quad C_{3,P} := \begin{bmatrix} -A_3 & -A_2 & I_n & 0 \\ -A_2 & -A_1 & 0 & -A_0 \\ I_n & 0 & 0 & 0 \\ 0 & -A_0 & 0 & 0 \end{bmatrix},$$

$$C_{h,P} := \left[\begin{array}{c|ccccc} T_h & \begin{matrix} 0 \\ E_{h-3} \end{matrix} & & & & \\ \hline \begin{matrix} 0 & E^{\mathcal{B}}_{h-3} \end{matrix} & \begin{matrix} D_{h-4} \\ E^{\mathcal{B}}_{h-5} \end{matrix} & E_{h-5} & & & \\ & & \ddots & & & \\ & & & D_4 & E_3 & \\ & & & E_3^{\mathcal{B}} & D_2 & E_1 \\ & & & & E_1^{\mathcal{B}} & D_0 \end{array}\right], \quad \text{for } 4 \le h < k \text{ even},$$

$$C_{h,P} := \left[\begin{array}{c|cccc|c} T_h & \begin{matrix} 0 \\ E_{h-3} \end{matrix} & & & & \\ \hline \begin{matrix} 0 & E^{\mathcal{B}}_{h-3} \end{matrix} & \begin{matrix} D_{h-4} \\ E^{\mathcal{B}}_{h-5} \end{matrix} & E_{h-5} & & & \\ & & \ddots & & & \\ & & & D_5 & E_4 & \\ & & & E_4^{\mathcal{B}} & D_3 & \begin{matrix} E_2 & 0 \end{matrix} \\ \hline & & & & \begin{matrix} E_2^{\mathcal{B}} \\ 0 \end{matrix} & T_1 \end{array}\right], \quad \text{for } 5 \le h < k \text{ odd},$$

$$\begin{aligned} C'_{0,P} &:= \emptyset, \\ C'_{h,P} &:= F_{h-1} \oplus F_{h-3} \oplus \cdots \oplus F_1, \quad \text{if } h \ge 2 \text{ even} \end{aligned}$$

and

$$C'_{h,P} := F_{h-1} \oplus F_{h-3} \oplus \cdots \oplus F_2 \oplus -A_0, \quad \text{for } 0 \le h < k \text{ odd}.$$

Observe that the size of $C_{h,P}$ is $n(h+1) \times n(h+1)$ and the size of $C'_{h,P}$ is $nh \times nh$. We also need block-sip matrices with blocks of size $n \times n$ as in (2.7) for different numbers of blocks. For brevity, all of them will be denoted by R, since their sizes will be clear from the context. One last ingredient is needed in the proof of Theorem 8.1: note that, according to Remark 4.5, it may happen that $M^P_{\mathbf{t}} = M^P_{\mathbf{t}'}$ for all $P(\lambda)$ for some $\mathbf{t} \neq \mathbf{t}'$. In this case, the tuples $\mathbf{t}$ and $\mathbf{t}'$ are said to be equivalent and this is denoted by $\mathbf{t} \sim \mathbf{t}'$.

THEOREM 8.1. *Let $P(\lambda)=\sum_{i=0}^{k} A_i\lambda^i$ be an $n\times n$ matrix polynomial of grade $k\geq 2$ and let h be an integer such that $0\leq h<k$. Then, the simple FPR $L_P(h,\emptyset,\emptyset)$ is equal to λL_1-L_0, where*

$$L_1 = RC_{k-h-1,-revP}R\oplus C'_{h,P} \quad \textit{and} \quad L_0 = RC'_{k-h-1,-revP}R\oplus C_{h,P}.$$

Proof. Let $\mathbf{w}_h$ and $\mathbf{w}_{k-h-1}$ denote the admissible tuples for h and $k-h-1$, respectively, and $\mathbf{c}_h$ and $\mathbf{c}_{k-h-1}$ denote the symmetric complements of $\mathbf{w}_h$ and $\mathbf{w}_{k-h-1}$, respectively. Let $\mathbf{v}_h=-k+\mathbf{w}_{k-h-1}$ and $\mathbf{d}_h=-k+\mathbf{c}_{k-h-1}$. Then, according to Definition 4.11,

$$L_P(h,\emptyset,\emptyset)=(\lambda M^P_{\mathbf{v}_h}-M^P_{\mathbf{w}_h})M^P_{\mathbf{d}_h,\mathbf{c}_h}. \tag{8.1}$$

The structure of the elementary matrices (4.3) and the ranges of the indices contained in $\mathbf{w}_h,\mathbf{v}_h,\mathbf{c}_h$, and $\mathbf{d}_h$ imply

$$M^P_{\mathbf{w}_h,\mathbf{c}_h}=I_{n(k-h-1)}\oplus B_1^h \quad \text{for some } B_1^h\in\mathbb{C}^{n(h+1)\times n(h+1)}, \tag{8.2}$$

$$M^P_{\mathbf{c}_h}=I_{n(k-h)}\oplus \widehat{B}_1^h \quad \text{for some } \widehat{B}_1^h\in\mathbb{C}^{nh\times nh}, \tag{8.3}$$

$$M^P_{\mathbf{v}_h,\mathbf{d}_h}=B_2^h\oplus I_{nh} \quad \text{for some } B_2^h\in\mathbb{C}^{n(k-h)\times n(k-h)}, \tag{8.4}$$

$$M^P_{\mathbf{d}_h}=\widehat{B}_2^h\oplus I_{n(h+1)} \quad \text{for some } \widehat{B}_2^h\in\mathbb{C}^{n(k-h-1)\times n(k-h-1)}. \tag{8.5}$$

From (8.3) and (8.5), we get that $M^P_{\mathbf{c}_h}$ and $M^P_{\mathbf{d}_h}$ commute. So, from (8.1), we obtain

$$L_1=M^P_{\mathbf{v}_h,\mathbf{d}_h}M^P_{\mathbf{c}_h}=B_2^h\oplus\widehat{B}_1^h, \tag{8.6}$$

$$L_0=M^P_{\mathbf{w}_h,\mathbf{c}_h}M^P_{\mathbf{d}_h}=\widehat{B}_2^h\oplus B_1^h\,. \tag{8.7}$$

Our next goal is to prove that

$$B_1^h=C_{h,P} \quad \text{and} \quad \widehat{B}_1^h=C'_{h,P}, \quad \text{for } 0\leq h<k. \tag{8.8}$$

For this purpose, we need to distinguish two cases: h even and h odd. We only prove (8.8) when h is even, since the proof for h odd is similar. So, let us assume that h is even and let us proceed by induction on even numbers. The results for $h=0$, $h=2$, and $h=4$ are established via direct computations of the left hand sides of the equations (8.2) and (8.3). More precisely, if $h=0$, then $\mathbf{w}_h=(0)$ and $\mathbf{c}_h=\emptyset$, and (8.8) follows in a straightforward way. If $h=2$, then $(\mathbf{w}_h,\mathbf{c}_h)=(1:2,0:1)$ and $\mathbf{c}_h=(1)$, and direct computations of $M^P_{\mathbf{w}_h,\mathbf{c}_h}$ and $M^P_{\mathbf{c}_h}$ show that

$$B_1^2=\begin{bmatrix}-A_2 & -A_1 & I_n\\ -A_1 & -A_0 & 0\\ I_n & 0 & 0\end{bmatrix}=C_{2,P} \quad \text{and} \quad \widehat{B}_1^2=\begin{bmatrix}-A_1 & I_n\\ I_n & 0\end{bmatrix}=C'_{2,P}.$$

Since $C_{h,P}$ has particular forms for the cases $h=0,2$, we prove (8.8) for $h=4$ as the base case of the induction. If $h=4$, then $(\mathbf{w}_h,\mathbf{c}_h)\sim(3:4,1:3,0:1)$ and $\mathbf{c}_h=(3,1)$, and again direct computations of $M^P_{\mathbf{w}_h,\mathbf{c}_h}$ and $M^P_{\mathbf{c}_h}$ show that

$$B_1^4=\left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{T_4}} & 0\\ & & E_1\\ \hline 0 & E_1^{\mathcal{B}} & D_0\end{array}\right]=C_{4,P} \quad \text{and} \quad \widehat{B}_1^4=F_3\oplus F_1=C'_{4,P}.$$

Now, we assume that (8.8) holds for some $h \geq 4$ even and prove it for $h+2$. Observe that $(\mathbf{w}_h, \mathbf{c}_h) \sim (h-1:h, h-3:h-1, \ldots, 3:5, 1:3, 0:1)$, which implies

$$(\mathbf{w}_{h+2}, \mathbf{c}_{h+2}) \sim (h+1:h+2, \mathbf{w}_h, \mathbf{c}_h, h+1) \quad \text{and} \quad \mathbf{c}_{h+2} = (h+1, \mathbf{c}_h). \tag{8.9}$$

The fact that $\widehat{B}_1^{h+2} = C'_{h+2,P}$ follows immediately from (8.3), the structure of M^P_{h+1} and the induction hypothesis $\widehat{B}_1^h = C'_{h,P}$. To prove the result for B_1^{h+2} requires more work. First note that from (8.9), we get

$$M^P_{\mathbf{w}_{h+2},\mathbf{c}_{h+2}} = M^P_{h+1:h+2}\, M^P_{\mathbf{w}_h,\mathbf{c}_h}\, M^P_{h+1}, \tag{8.10}$$

and next observe that a direct computation shows that

$$M^P_{h+1:h+2} = I_{n(k-h-3)} \oplus Q \oplus I_{nh}, \quad \text{where} \quad Q = \begin{bmatrix} -A_{h+2} & I_n & 0 \\ -A_{h+1} & 0 & I_n \\ I_n & 0 & 0 \end{bmatrix}. \tag{8.11}$$

On the other hand, the induction hypothesis and (8.2) yield $M^P_{\mathbf{w}_h,\mathbf{c}_h} = I_{n(k-h-1)} \oplus C_{h,P}$ and (4.3) implies $M^P_{h+1} = I_{n(k-h-2)} \oplus F_{h+1} \oplus I_{nh}$, which, combined with (8.10), (8.11), and (8.2) for $h+2$, gives

$$B_1^{h+2} = (Q \oplus I_{nh})(I_{2n} \oplus C_{h,P}) \left(I_n \oplus \begin{bmatrix} -A_{h+1} & I_n \\ I_n & 0 \end{bmatrix} \oplus I_{nh} \right). \tag{8.12}$$

A direct multiplication shows that

$$(Q \oplus I_{nh})(I_{2n} \oplus C_{h,P}) = \left[\begin{array}{c|cccccc} \widetilde{Q} & \begin{matrix} 0 \\ E_{h-1} \end{matrix} & & & & & \\ \hline 0 \ (E^{\mathcal{B}}_{h-1} R) & \begin{matrix} D_{h-2} \\ E^{\mathcal{B}}_{h-3} \end{matrix} & \begin{matrix} E_{h-3} \\ D_{h-4} \\ E^{\mathcal{B}}_{h-5} \end{matrix} & E_{h-5} & & & \\ & & & \ddots & & & \\ & & & & \begin{matrix} D_4 \\ E^{\mathcal{B}}_3 \end{matrix} & \begin{matrix} E_3 \\ D_2 \\ E^{\mathcal{B}}_1 \end{matrix} & \begin{matrix} \\ E_1 \\ D_0 \end{matrix} \end{array}\right],$$

where the first three columns of the previous matrix are

$$\left[\begin{array}{ccc} -A_{h+2} & I_n & 0 \\ -A_{h+1} & 0 & -A_h \\ I_n & 0 & 0 \\ \hline 0 & 0 & -A_{h-1} \\ 0 & 0 & I_n \\ \hline 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{array}\right].$$

From here and (8.12), we get $B_1^{h+2} = C_{h+2,P}$ immediately and the inductive proof of (8.8) for h even is completed.

Now, we prove

$$B_2^h = RC_{k-h-1,-revP}R \quad \text{and} \quad \widehat{B}_2^h = RC'_{k-h-1,-revP}R. \tag{8.13}$$

From (8.8), (8.2), and (8.3), we have

$$M^P_{\mathbf{w}_h,\mathbf{c}_h} = I_{n(k-h-1)} \oplus C_{h,P}, \text{ and } M^P_{\mathbf{c}_h} = I_{n(k-h)} \oplus C'_{h,P}, \text{ for } 0 \le h < k, \tag{8.14}$$

which hold for any matrix polynomial $P(\lambda)$. We also have $M^P_i = RM^{-rev(P)}_{k+i}R, \quad i \in \{-k : -1\}$, which follows from the structure of M^P_i in (4.3), and $R^2 = I$. These two facts imply

$$M^P_{\mathbf{v}_h,\mathbf{d}_h} = RM^{-revP}_{\mathbf{w}_{k-h-1},\mathbf{c}_{k-h-1}}R \quad \text{and} \quad M^P_{\mathbf{d}_h} = RM^{-revP}_{\mathbf{c}_{k-h-1}}R. \tag{8.15}$$

Applying (8.14) to $M^{-revP}_{\mathbf{w}_{k-h-1},\mathbf{c}_{k-h-1}}$ and $M^{-revP}_{\mathbf{c}_{k-h-1}}$, which requires the use of $k-h-1$ instead of h, and taking (8.4), (8.5), and (8.15) into account, (8.13) follows.

Finally, Theorem 8.1 can be deduced from (8.6), (8.7), (8.8), and (8.13). □

EXAMPLE 8.2. *This example illustrates Theorem 8.1 for $k = 10$ and $h = 4$. In this case, $L_P(4, \emptyset, \emptyset) = \lambda L_1 - L_0$, where*

$$L_1 = \left[\begin{array}{cccccc|cccc}
0 & 0 & A_{10} & & & & & & & \\
0 & 0 & 0 & 0 & I_n & & & & & \\
A_{10} & 0 & A_9 & 0 & A_8 & & & & & \\
 & 0 & 0 & 0 & 0 & I_n & & & & \\
 & I_n & A_8 & 0 & A_7 & A_6 & & & & \\
 & & & I_n & A_6 & A_5 & & & & \\
\hline
 & & & & & & -A_3 & I_n & & \\
 & & & & & & I_n & 0 & & \\
 & & & & & & & & -A_1 & I_n \\
 & & & & & & & & I_n & 0
\end{array}\right],$$

$$L_0 = \left[\begin{array}{ccccc|ccccc}
A_{10} & & & & & & & & & \\
 & 0 & I_n & & & & & & & \\
 & I_n & A_8 & & & & & & & \\
 & & & 0 & I_n & & & & & \\
 & & & I_n & A_6 & & & & & \\
\hline
 & & & & & -A_4 & -A_3 & I_n & & \\
 & & & & & -A_3 & -A_2 & 0 & -A_1 & I_n \\
 & & & & & I_n & 0 & 0 & 0 & 0 \\
 & & & & & & -A_1 & 0 & -A_0 & 0 \\
 & & & & & & I_n & 0 & 0 & 0
\end{array}\right].$$

Recall that, according to Theorem 7.1, if $P(\lambda)$ is Hermitian and A_{10} is nonsingular, then $L_P(4, \emptyset, \emptyset)$ is a Hermitian strong linearization that preserves the sign characteristic of $P(\lambda)$.

9. Conclusions. In this paper we show that the Hermitian strong linearizations of a Hermitian matrix polynomial $P(\lambda)$ of degree k with nonsingular leading coefficient that preserve its sign characteristic are precisely the pencils *congruent to the kth pencil in the standard basis for $\mathbb{DL}(P)$. Additionally, we have identified several classes of such strong linearizations of $P(\lambda)$ that can be easily constructed from the coefficients of $P(\lambda)$. All these linearizations are related to Fiedler pencils and belong either to the family of generalized Fiedler pencils [9] or to the family of generalized

Fiedler pencils with repetition [10]. Particularly relevant examples include the block-tridiagonal linearizations in [3] and [23, 24, 25], some pencils in the standard basis of $\mathbb{DL}(P)$, and some simple Fiedler pencils with repetition. The tools developed in this work may allow us to identify in the future other classes of sign characteristic preserving linearizations of Hermitian matrix polynomials. In a work in progress, we intend to give a full characterization of the pencils in the space $\mathbb{DL}(P)$ and in the family of HGFPR which preserve the sign characteristic of $P(\lambda)$ (note that here we only studied the pencils in the standard basis of $\mathbb{DL}(P)$ and we only gave necessary conditions for the sign characteristic to be preserved). In addition, future research will include the study of the conditioning and the backward errors of the eigenvalues of the classes of linearizations identified in this paper.

We also note that, in a very recent paper [30], the classical definition of sign characteristic of a Hermitian matrix polynomial has been extended to general analytic Hermitian matrix functions, which include the case of general (regular or singular) Hermitian matrix polynomials in the complex and real fields. An interesting problem, which we plan to address in a future work, is to consider matrix polynomials with singular leading matrix coefficient (such matrix polynomials can be either singular or regular with infinite elementary divisors) and extend the results in this paper by considering this more general definition of sign characteristic. We note, however, that an approach different from the one considered here should be taken as the last pencil $D_k(\lambda, P)$ in the standard basis of the vector space $\mathbb{DL}(P)$, which plays a crucial role in our work, is not a linearization of $P(\lambda)$ anymore when $P(\lambda)$ has singular leading coefficient.

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