ON THE SIGN CHARACTERISTIC OF HERMITIAN LINEARIZATIONS IN $\mathbb{DL}(P)$

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Abstract. The computation of eigenvalues and eigenvectors of matrix polynomials is an important, but difficult, problem. The standard approach to solve this problem is to use linearizations, which are matrix polynomials of degree 1 that share the eigenvalues of $P(\lambda)$.

Hermitian matrix polynomials and their real eigenvalues are of particular interest in applications. Attached to these eigenvalues is a set of signs called the sign characteristic. From both a theoretical and a practical point of view, it is important to be able to recover the sign characteristic of a Hermitian linearization of $P(\lambda)$ from the sign characteristic of $P(\lambda)$.

In this paper, for a Hermitian matrix polynomial $P(\lambda)$ with nonsingular leading coefficient, we describe, in terms of the sign characteristic of $P(\lambda)$, the sign characteristic of the Hermitian linearizations in the vector space $\mathbb{DL}(P)$ (Mackey, Mackey, Mehl and Mehrmann, 2006). In particular, we identify the Hermitian linearizations in $\mathbb{DL}(P)$ that preserve the sign characteristic of $P(\lambda)$. We also provide a description of the sign characteristic of the Hermitian linearizations of $P(\lambda)$ in the family of generalized Fiedler pencils with repetition (Bueno, Dopico, Furtado and Rychnovsky, 2015).

Key words. Hermitian matrix polynomial, Hermitian linearization, sign characteristic, $\mathbb{DL}(P)$, generalized Fiedler pencil with repetition.

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1. Introduction.

Let

$$P(\lambda) = \sum_{i=0}^{k} A_i \lambda^i \tag{1.1}$$

be a matrix polynomial of degree k with $A_i \in \mathbb{C}^{n\times n}$. In this paper we assume that $P(\lambda)$ is *regular*, that is, $\det(P(\lambda))$ is not identically zero.

The *polynomial eigenvalue problem* (PEP)

$$P(\lambda)x = 0 \tag{1.2}$$

arises in many areas such as control theory, signal processing, and vibration analysis. The solutions λ and x of (1.2) are called the *(finite) eigenvalues* and the *(right) eigenvectors associated with* λ, respectively, of the matrix polynomial $P(\lambda)$. The standard technique to solve the PEP is to replace $P(\lambda)$ by a linearization of $P(\lambda)$ and solve the corresponding linear eigenvalue problem.

A *linearization* of the matrix polynomial $P(\lambda)$ is a pencil $L(\lambda) = \lambda L_1 - L_0$ of size $nk \times nk$ such that

$$U(\lambda)\, L(\lambda)\, V(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{n(k-1)} \end{bmatrix},$$

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for some matrix polynomials $U(\lambda)$ and $V(\lambda)$ with constant nonzero determinant. (For a positive integer r, I_r denotes the identity matrix of size $r \times r$.) It is well known that linearizations of a matrix polynomial $P(\lambda)$ have the same elementary divisors and, in particular, the same eigenvalues as $P(\lambda)$. We observe that we are using the classical definition of linearization of a matrix polynomial (see, for example, [6, 8], where regular matrix polynomials are considered). Other more general definitions of linearization have been introduced recently [4].

In many applications, the matrix polynomial $P(\lambda)$ arises with some structure, such as Hermitian, symmetric, skew-symmetric, palindromic, etc. In order to preserve the properties of the eigenvalues imposed by that structure, it is convenient to find linearizations of $P(\lambda)$ with the same structure, if such linearizations exist. In this paper we will focus on the important class of Hermitian matrix polynomials. A matrix polynomial $P(\lambda)$ as in (1.1) is said to be *Hermitian* if $A_i = A_i^*$, $i = 1, \ldots, k$, where A_i^* denotes the conjugate transpose of A_i.

Hermitian linearizations of Hermitian matrix polynomials $P(\lambda)$ have been developed in the literature. An important class of such linearizations is contained in the vector space $\mathbb{DL}(P)$ introduced in [13]. In [2] a new infinite class of Hermitian pencils, called Hermitian generalized Fiedler pencils with repetition (GFPR), was constructed containing the finite class of Hermitian Fiedler pencils with repetition introduced in [15], and, in particular, the pencils in the standard basis of $\mathbb{DL}(P)$.

An important feature of a Hermitian matrix polynomial $P(\lambda)$ is its sign characteristic. Here we consider the classical definition of the sign characteristic given in [6, 8], which assumes that $P(\lambda)$ has a nonsingular leading coefficient and assigns a set of signs to the (finite) elementary divisors of $P(\lambda)$. We note that, in a recent still unpublished paper [14], an extension of the classical definition of sign characteristic to infinite elementary divisors as well as to singular matrix polynomials (that is, matrix polynomials that are not regular) is developed. An interesting problem, that we plan to address in a future work, is generalizing the results in this paper by considering this extended definition of sign characteristic.

Among other relevant applications, the sign characteristic plays an important role in perturbation theory of Hermitian matrix polynomials. It is well known that a Hermitian linearization of a Hermitian matrix polynomial $P(\lambda)$ may have a sign characteristic different from the one of $P(\lambda)$. In general, it is convenient to use Hermitian linearizations with the same sign characteristic as $P(\lambda)$. In [1] it was shown, under some restrictions on the eigenvalues, that the last pencil in the standard basis of $\mathbb{DL}(P)$, which is known to be a linearization of $P(\lambda)$ with nonsingular leading coefficient, preserves the sign characteristic of $P(\lambda)$. This result was generalized in [3] by relaxing the restrictions on the eigenvalues of $P(\lambda)$. Based on this result, a class of Hermitian linearizations in the family of Hermitian GFPR preserving the sign characteristic of $P(\lambda)$ was identified in [3]. We note, however, that in some circumstances, it is important that a linearization of $P(\lambda)$ preserves some additional structure of $P(\lambda)$ associated with relevant spectral properties and this may not be possible by considering a linearization with the same sign characteristic as $P(\lambda)$ [9]. In such a case, it is important to know how the sign characteristic of the linearization changes with respect to the sign characteristic of $P(\lambda)$.

In this paper, given a Hermitian matrix polynomial with nonsingular leading coefficient, we describe the sign characteristic of the Hermitian linearizations in $\mathbb{DL}(P)$ in terms of the sign characteristic of $P(\lambda)$ and, in particular, we identify those linearizations that preserve the sign characteristic of $P(\lambda)$. As a consequence of this result, we give a similar description of the sign characteristic of all the Hermitian GFPR linearizations of $P(\lambda)$.

This paper is organized as follows. In Section 2 we introduce some basic concepts and results from the general theory of matrix polynomials, define the sign characteristic of a Hermitian matrix polynomial with nonsingular leading coefficient and give a related result that will be used in the paper. In Section 3, we review the vector space $\mathbb{DL}(P)$, as well as some related concepts and results, and identify the subfamily of pencils that are Hermitian when $P(\lambda)$ is. In Section 4, we state the main result of this paper, Theorem 4.1, which provides a description of the sign characteristic of the Hermitian linearizations in $\mathbb{DL}(P)$ of a matrix polynomial $P(\lambda)$ in terms of the sign characteristic of $P(\lambda)$, when $P(\lambda)$ is Hermitian with nonsingular leading coefficient. Also, as a consequence of this result, we describe the sign characteristic of the Hermitian GFPR linearizations of $P(\lambda)$. The rest of the paper is dedicated to the proof of Theorem 4.1. In Section 5 we construct Jordan chains of the pencils in $\mathbb{DL}(P)$ from Jordan chains of $P(\lambda)$. In Section 6 we describe the structure of a certain matrix $H(X, J, v)$ constructed from a pencil in $\mathbb{DL}(P)$ and its Jordan chains, that has a key role in obtaining our main results. In Section 7 we study the equation $Q^*\mathcal{L}Q = H$, where H is a generic matrix with the same structure as $H(X, J, v)$ and

$$\mathcal{L} = t_1\mathcal{R}_{s_1} \oplus \cdots \oplus t_l\mathcal{R}_{s_l},$$

in which $t_i = \pm 1$ are certain specific signs and $\mathcal{R}_s$ denotes the matrix obtained from the identity of size s by reversing the order of the rows. The main result in this section is Theorem 7.8, which is used in the proof of Theorem 4.1 and may have interest by itself. In Section 8 we prove Theorem 4.1. Finally, in Section 9, we summarize our main contributions in this paper and identify some open problems.

2. Basic concepts. In this section we introduce some important concepts from the theory of matrix polynomials and define the sign characteristic of a Hermitian matrix polynomial $P(\lambda)$ with nonsingular leading coefficient.

We will use the following notation: for $j \geq 0$, $P^{(j)}(\lambda)$ denotes the j-th derivative of $P(\lambda)$ with respect to λ. For the first derivative, we also use the standard notation: $P'(\lambda)$.

A sequence of vectors $\{x_1, \ldots, x_r\}$ in $\mathbb{C}^n$ is called a *Jordan chain of length* r of a regular matrix polynomial $P(\lambda)$, at the finite eigenvalue λ_0, if $x_1 \neq 0$ and

$$\begin{aligned} P(\lambda_0)x_1 &= 0 \\ P'(\lambda_0)x_1 + P(\lambda_0)x_2 &= 0 \\ &\vdots \\ \sum_{j=0}^{r-1} \frac{1}{j!} P^{(j)}(\lambda_0)x_{r-j} &= 0. \end{aligned}$$

A Jordan chain $\{x_1, \ldots, x_r\}$ is said to be *maximal* if there is no vector $x_{r+1} \in \mathbb{C}^n$ such that $\{x_1, \ldots, x_{r+1}\}$ is a Jordan chain. Note that the vectors in a Jordan chain of a matrix polynomial may be zero and they are not necessarily linearly independent.

The concept of Jordan chain of a matrix polynomial extends the well-known concept of Jordan chain of a constant matrix A, or equivalently, the concept of Jordan chain of a monic matrix pencil $\lambda I - A$.

In the rest of this section, we assume that $P(\lambda)$ is an $n \times n$ matrix polynomial of degree k as in (1.1) with nonsingular A_k. We associate to $P(\lambda)$ the following $nk \times nk$ matrices:

$$C_P = \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_n \\ -A_k^{-1}A_0 & -A_k^{-1}A_1 & -A_k^{-1}A_2 & \cdots & -A_k^{-1}A_{k-1} \end{bmatrix}, \quad B_P = \begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ A_2 & \vdots & \iddots & \\ \vdots & A_k & & \\ A_k & & & 0 \end{bmatrix}.$$

The matrix C_P is called *the companion matrix of* $P(\lambda)$. It is well-known that the pencil $\lambda I - C_P$ is a linearization of $P(\lambda)$ [6, 8] and, therefore, the matrix C_P has the same finite elementary divisors as $P(\lambda)$.

Clearly, B_P is nonsingular as A_k is. Moreover, if $P(\lambda)$ is Hermitian, then B_P is Hermitian and $C_P^* = B_P C_P B_P^{-1}$.

Observe that, if $L(\lambda) = \lambda L_1 - L_0$ is a pencil with L_1 nonsingular, then

$$C_L = L_1^{-1}L_0 \text{ and } B_L = L_1.$$

In what follows, given $\lambda \in \mathbb{C}$, we denote by $J_l(\lambda)$ the $l \times l$ Jordan block associated with the eigenvalue λ.

Suppose that S is a nonsingular matrix such that

$$J := S^{-1}C_P S = J_{l_1}(\lambda_1) \oplus \cdots \oplus J_{l_r}(\lambda_r),$$

is in Jordan form, where $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ are the eigenvalues of $P(\lambda)$. It can be easily seen that, for the partition of S

$$S = [S_1 \ \cdots \ S_r], \tag{2.1}$$

where S_i, $i = 1, \ldots, r$, has l_i columns, the columns of S_i form a maximal Jordan chain for C_P associated with λ_i. Let

$$X = [I_n \ 0 \ \cdots \ 0]S. \tag{2.2}$$

Partitioning $X = [X_1 \ \cdots \ X_r]$ according to the partition of S given in (2.1), the columns of each X_i form maximal Jordan chains for $P(\lambda)$ associated with λ_i [5, 7]. The pair (X, J) is called a *Jordan pair* for $P(\lambda)$.

Given a Jordan pair (X, J), the matrix S can be recovered from (X, J) as follows [6, Proposition 12.1.1]:

$$S = \begin{bmatrix} X \\ XJ \\ \vdots \\ XJ^{k-1} \end{bmatrix}. \tag{2.3}$$

In particular, if $P(\lambda)$ is a matrix pencil, then $S = X$.

The matrix (2.3) constructed from given X and J will be called the *(X, J)-matrix*. By definition of Jordan pair, if (X, J) is a Jordan pair for $P(\lambda)$ and S is the (X, J)-matrix, then $J = S^{-1}C_P S$.

In the rest of this section, we assume that $P(\lambda)$ is Hermitian with nonsingular leading coefficient. Note that, when $P(\lambda)$ is Hermitian, the spectrum of $P(\lambda)$ is symmetric with respect to the real axis. Moreover, the eigenvalues of $P(\lambda)$ are either real or occur in conjugate pairs [6, Proposition 4.2.3 and Section 5.1].

Throughout the paper, we denote by $\mathcal{R}_m$, or simply $\mathcal{R}$ (if there is no ambiguity with respect to the size), the $m \times m$ matrix:

$$\mathcal{R}_m := \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}. \tag{2.4}$$

Definition 2.1. *Let J be a matrix in Jordan form:*

$$\begin{aligned} J = & J_{l_1}(\lambda_1) \oplus \cdots \oplus J_{l_\alpha}(\lambda_\alpha) \\ & \oplus \left(J_{l_{\alpha+1}}(\lambda_{\alpha+1}) \oplus J_{l_{\alpha+1}}(\overline{\lambda_{\alpha+1}})\right) \oplus \cdots \oplus \left(J_{l_\beta}(\lambda_\beta) \oplus J_{l_\beta}(\overline{\lambda_\beta})\right), \end{aligned} \tag{2.5}$$

where $\lambda_1, \ldots, \lambda_\alpha \in \mathbb{R}$ and $\lambda_{\alpha+1}, \ldots, \lambda_\beta$ are nonreal complex numbers on the upper half-plane. Let $\epsilon = \{\epsilon_1, \ldots, \epsilon_\alpha\}$ be an ordered set of α signs ± 1. Then, we denote

$$P_{\epsilon,J} := \epsilon_1 \mathcal{R}_{l_1} \oplus \cdots \oplus \epsilon_\alpha \mathcal{R}_{l_\alpha} \oplus \mathcal{R}_{2l_{\alpha+1}} \oplus \cdots \oplus \mathcal{R}_{2l_\beta}.$$

and call ϵ a set of signs associated with J.

Note that the direct summand in $P_{\epsilon,J}$ corresponding to the nonreal eigenvalues of J is completely determined by J, that is, it does not depend on ϵ.

Theorem 2.2. *[6, Theorem 5.1.1] Let $P(\lambda)$ be an $n \times n$ Hermitian matrix polynomial as in (1.1) with nonsingular A_k. Let $\lambda_1, \ldots, \lambda_\alpha$ be the real eigenvalues and $\lambda_{\alpha+1}, \ldots, \lambda_\beta$ be the nonreal eigenvalues of $P(\lambda)$ from the upper half-plane. Then, there exists a nonsingular matrix S such that $J := S^{-1} C_P S$ is as in (2.5) and $S^* B_P S = P_{\epsilon,J}$ for some ordered set of signs $\epsilon = \{\epsilon_1, \ldots, \epsilon_\alpha\}$. Moreover, the set ϵ is unique (up to permutation of signs corresponding to identical Jordan blocks in J).*

Definition 2.3. *Let $P(\lambda)$ be an $n \times n$ Hermitian matrix polynomial with nonsingular leading coefficient and let ϵ be the set of signs given by Theorem 2.2. The set ϵ is called* the sign characteristic of $P(\lambda)$. *Moreover, we call the pair $(J, P_{\epsilon,J})$ in Theorem 2.2 a* canonical pair for $P(\lambda)$ *and we call the Jordan pair (X, J) a* reducing Jordan pair *associated with $P(\lambda)$, where X is as in (2.2) with S as in Theorem 2.2.*

Note that the sign characteristic of $P(\lambda)$ attaches a sign to each elementary divisor of $P(\lambda)$ associated with a real eigenvalue. When we fix an order for the elementary divisors of $P(\lambda)$ associated with the real eigenvalues, namely, $(\lambda - \lambda_1)^{s_1}, \ldots, (\lambda - \lambda_\alpha)^{s_\alpha}$, and say that $P(\lambda)$ has sign characteristic $\epsilon_1, \ldots, \epsilon_\alpha$, we mean that the sign associated with the elementary divisor $(\lambda - \lambda_i)^{s_i}$ is ϵ_i, $i = 1, \ldots, \alpha$.

Clearly, if J' and $P_{\epsilon',J'}$ are obtained from J and $P_{\epsilon,J}$ by a simultaneous block permutation similarity, then $(J', P_{\epsilon',J'})$ is still a canonical pair for $P(\lambda)$.

The next result gives necessary and sufficient conditions for a pair $(J, P_{\epsilon,J})$ to be a canonical pair for a matrix polynomial.

Proposition 2.4. *Let $P(\lambda)$ be an $n \times n$ Hermitian matrix polynomial as in (1.1) with nonsingular A_k. Let (X, J) be a Jordan pair for $P(\lambda)$ and let ϵ be an ordered set of signs associated with J. Then, $(J, P_{\epsilon,J})$ is a canonical pair for $P(\lambda)$ if and only if there exists a nonsingular matrix Q such that*

$$J = Q^{-1} J Q \quad \textit{and} \quad Q^* P_{\epsilon,J} Q = S^* B_P S$$

where S is the (X, J)-matrix.

Proof. We have

$$Q^* P_{\epsilon,J} Q = S^* B_P S \Leftrightarrow P_{\epsilon,J} = Q^{-*} S^* B_P S Q^{-1} \Leftrightarrow P_{\epsilon,J} = W^* B_P W$$
$$J = Q^{-1} J Q \Leftrightarrow S^{-1} C_P S = Q^{-1} J Q \Leftrightarrow W^{-1} C_P W = J,$$

where $W = SQ^{-1}$. Note that $J = S^{-1} C_P S$, as (X, J) is a Jordan pair for $P(\lambda)$ and S is the (X, J)-matrix. ☐

3. Hermitian linearizations in $\mathbb{DL}(P)$. Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree k as in (1.1). In what follows, let

$$\Lambda(\lambda) := [\lambda^{k-1}\, \lambda^{k-2}\, \cdots\, \lambda\, 1]^T \tag{3.1}$$

and let $\otimes$ denote the Kronecker product.

The following vector spaces of $nk \times nk$ pencils were defined in [13]:

$$\mathbb{L}_1(P) := \left\{ L(\lambda) = \lambda X + Y \,:\, L(\lambda)(\Lambda(\lambda) \otimes I_n) = v \otimes P(\lambda),\, v \in \mathbb{C}^k \right\},$$
$$\mathbb{L}_2(P) := \left\{ L(\lambda) = \lambda X + Y \,:\, (\Lambda(\lambda)^T \otimes I_n) L(\lambda) = w^T \otimes P(\lambda),\, w \in \mathbb{C}^k \right\}.$$

The intersection of these two vector spaces is the vector space

$$\mathbb{DL}(P) := \mathbb{L}_1(P) \cap \mathbb{L}_2(P).$$

It is well known that, for a pencil in $\mathbb{DL}(P)$ associated with a vector v in $\mathbb{L}_1(P)$ and a vector w in $\mathbb{L}_2(P)$, we have $v = w$. We denote this pencil by $D(P, v)$. The corresponding vector v is called the *ansatz vector* of $D(P, v)$. The space $\mathbb{DL}(P)$ consists of block-symmetric pencils, most of which are linearizations of $P(\lambda)$ [13]. Moreover, $\mathbb{DL}(P)$ is a vector space of dimension k, the degree of $P(\lambda)$. A basis of $\mathbb{DL}(P)$, called the *standard basis*, is formed by the pencils $D(P, e_i)$, $i = 1, \ldots, k$, where e_i denotes the ith column of the $k \times k$ identity matrix I_k.

As in [1], we associate to the pencil $D(P, v) \in \mathbb{DL}(P)$ the polynomial $p(\lambda; v)$ defined in terms of the ansatz vector $v = [v_1, \ldots, v_k]^T \in \mathbb{C}^k$ and given by

$$p(\lambda; v) = \Lambda(\lambda)^T v = \lambda^{k-1} v_1 + \lambda^{k-2} v_2 + \cdots + \lambda v_{k-1} + v_k. \tag{3.2}$$

We call $p(\lambda; v)$ the *v-polynomial*. Next, a characterization of the pencils in $\mathbb{DL}(P)$ which are linearizations of $P(\lambda)$ is given in terms of the v-polynomial.

THEOREM 3.1. *[13] Suppose that $P(\lambda)$ is a regular matrix polynomial and let $0 \neq v \in \mathbb{C}^k$. Then, $D(P, v)$ is a linearization of $P(\lambda)$ if and only if no root of the v-polynomial $p(\lambda; v)$ is an eigenvalue of $P(\lambda)$, where, by convention, $p(\lambda; v)$ has a root at ∞ whenever $v_1 = 0$.*

The next result gives a relationship between the (right and left) eigenvectors of $P(\lambda)$ and the (right and left) eigenvectors of any linearization of $P(\lambda)$ in $\mathbb{DL}(P)$.

THEOREM 3.2. *(Eigenvector Recovery Property) [13]Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree k, and let $v \in \mathbb{C}^k \backslash \{0\}$. Let $\lambda_0 \in \mathbb{C}$ be an eigenvalue of $P(\lambda)$. Then, x (resp. y) is a right (resp. left) eigenvector for $P(\lambda)$ associated with λ_0 if and only if $\Lambda(\lambda_0) \otimes x$ (resp. $\overline{\Lambda(\lambda_0)} \otimes y$) is a right (resp. left) eigenvector for $D(P, v)$ associated with λ_0. If, in addition, $P(\lambda)$ is regular and $D(P, v)$ is a linearization of $P(\lambda)$, then every right (resp. left) eigenvector of $D(P, v)$ with finite eigenvalue λ_0 is of the form $\Lambda(\lambda_0) \otimes x$ (resp. $\overline{\Lambda(\lambda_0)} \otimes y$) for some right (resp. left) eigenvector x (resp. y) of $P(\lambda)$.*

We now focus on the case in which $P(\lambda)$ is a Hermitian matrix polynomial of degree k. For such a $P(\lambda)$, we denote by $\mathbb{H}(P)$ the subset of $\mathbb{DL}(P)$ that consists of its Hermitian pencils, that is,

$$\mathbb{H}(P) = \{L(\lambda) \in \mathbb{DL}(P) \,:\, L(\lambda) \text{ is Hermitian}\}.$$

In fact, the pencils in $\mathbb{H}(P)$ are those whose ansatz vector v has real entries, that is, $v \in \mathbb{R}^k$ [10, Lemma 6.1].

One of the main goals of this paper is to express the sign characteristic of a linearization of $P(\lambda)$ in $\mathbb{H}(P)$ in terms of the sign characteristic of $P(\lambda)$, when $P(\lambda)$ is Hermitian with nonsingular leading coefficient. As a corollary, we will determine the linearizations in $\mathbb{H}(P)$ that preserve its sign characteristic.

4. Main results.

We give next the main result of this paper which describes the sign characteristic of a linearization $D(P, v) \in \mathbb{H}(P)$ of a Hermitian matrix polynomial $P(\lambda)$ with nonsingular leading coefficient in terms of the sign characteristic of $P(\lambda)$. The proof of this result will be presented in Section 8.

Given a real number $b \neq 0$, we denote $sign(b) = 1$ if $b > 0$ and $sign(b) = -1$ if $b < 0$.

THEOREM 4.1. *Let $P(\lambda)$ be an $n \times n$ Hermitian matrix polynomial as in (1.1) with nonsingular A_k. Let $L(\lambda) = D(P, v) \in \mathbb{H}(P)$ be a linearization of $P(\lambda)$. Let $(\lambda - \lambda_1)^{s_1}, \ldots, (\lambda - \lambda_\alpha)^{s_\alpha}$ be an ordered list of the elementary divisors of $P(\lambda)$ associated with the real eigenvalues $\lambda_1, \ldots, \lambda_\alpha$, and let $\epsilon_1, \ldots, \epsilon_\alpha$ be the corresponding signs in the sign characteristic of $P(\lambda)$. Then, the sign characteristic of $D(P, v)$ is*

$$sign(p(\lambda_1; v))\epsilon_1, \ldots, sign(p(\lambda_\alpha; v))\epsilon_\alpha.$$

In the next result, which is an immediate consequence of Theorem 4.1, we characterize the linearizations in $\mathbb{H}(P)$ that preserve the sign characteristic of $P(\lambda)$.

COROLLARY 4.2. *Let $P(\lambda)$ be a Hermitian matrix polynomial as in (1.1) with nonsingular A_k. Let $D(P, v) \in \mathbb{H}(P)$ be a linearization of $P(\lambda)$. Then, $D(P, v)$ preserves the sign characteristic of $P(\lambda)$ if and only if, for each eigenvalue λ_i of $P(\lambda)$ such that $p(\lambda_i; v) < 0$, the following conditions hold:*

- *the number of negative signs and the number of positive signs in the sign characteristic of $P(\lambda)$ corresponding to the elementary divisors of $P(\lambda)$ associated with λ_i coincide;*
- *the elementary divisors associated with positive signs can be paired with the elementary divisors associated with negative signs in such a way that the paired elementary divisors are identical.*

From the previous corollary, we obtain a simple sufficient condition for a linearization in $\mathbb{H}(P)$ to preserve the sign characteristic of $P(\lambda)$.

COROLLARY 4.3. *Let $P(\lambda)$ be a Hermitian matrix polynomial as in (1.1) with nonsingular A_k and let $D(P, v) \in \mathbb{H}(P)$ be a linearization of $P(\lambda)$. If $p(\lambda_0; v) > 0$ for all real eigenvalues λ_0 of $P(\lambda)$, then $D(P, v)$ preserves the sign characteristic of $P(\lambda)$.*

For a matrix polynomial $P(\lambda)$ of degree k as in (1.1), we described, in [2], an infinite family of pencils, called *generalized Fiedler pencils with repetition* (GFPR), most of which are linearizations of $P(\lambda)$. An infinite subfamily of the GFPR formed by block-symmetric pencils was also identified. A pencil in this subfamily, denoted by $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$, is determined by a parameter h, with $0 \leq h < k$, some sets of indices $\mathbf{t}_w \subseteq \{0, \ldots, h-2\}$ and $\mathbf{t}_v \subseteq \{-k, \ldots, -h-3\}$, and some sequences Z_w, Z_v of $n \times n$ matrices assigned to the tuples $\mathbf{t}_w$ and $\mathbf{t}_v$. Since a formal description of these pencils is very technical, we omit it here. For a detailed description of the class of block-symmetric GFPR, see [2] (for a summarized description, see [3, Section 5]). In these references, it is shown that, for each $i = 1, \ldots, k$, the pencil $D(P, e_i)$ in the standard basis of the vector space $\mathbb{DL}(P)$ is a block-symmetric GFPR associated with the parameter $h = k - i$. When $P(\lambda)$ is a Hermitian matrix polynomial and the matrices in Z_w and Z_v are Hermitian, the block-symmetric GFPR $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ is also Hermitian.

Assume that $P(\lambda)$ is Hermitian with nonsingular leading coefficient. As a consequence of Theorem 4.1, we next give a complete description of the sign characteristic of the Hermitian GFPR linearizations $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ of $P(\lambda)$ in terms of the sign characteristic of $P(\lambda)$. We note that, when h is even, this description is given in [3, Theorem 8.1].

In [3, Lemma 6.1], it is shown that $*$-congruent Hermitian linearizations of $P(\lambda)$ have the same sign characteristic. Moreover, it is shown in this reference that, for h even, any Hermitian GFPR linearization $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ of $P(\lambda)$ is $*$-congruent to $D(P, e_k)$ (see the proof of [3, Theorem 8.1]), which implies that $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ and $D(P, e_k)$ have the same sign characteristic [3, Theorem 8.1]. On the other hand, when h is odd, any Hermitian GFPR linearization $L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$ of $P(\lambda)$ is $*$-congruent to $D(P, e_{k-1})$ (see [3, Remark 8.5]), and, thus, both have the same sign characteristic. Therefore, from Theorem 4.1, and taking into account that the v-polynomial $p(\lambda; v)$ associated with $D(P, e_k)$ and $D(P, e_{k-1})$ is 1 and λ, respectively, we obtain the following result.

Theorem 4.4. *Let $P(\lambda)$ be a Hermitian matrix polynomial of degree k as in (1.1) with invertible A_k. Let $(\lambda - \lambda_1)^{s_1}, \ldots, (\lambda - \lambda_\alpha)^{s_\alpha}$ be an ordered list of the elementary divisors of $P(\lambda)$ associated with the real eigenvalues and $\epsilon_1, \ldots, \epsilon_\alpha$ be the corresponding signs in the sign characteristic of $P(\lambda)$. Let $L(\lambda) = L_P(h, \mathbf{t}_w, \mathbf{t}_v, Z_w, Z_v)$, with $0 \leq h < k$, be a Hermitian GFPR linearization of $P(\lambda)$. Then,*

- *if h is even, $P(\lambda)$ and $L(\lambda)$ have the same sign characteristic;*
- *if h is odd, the sign characteristic of $L(\lambda)$ is given by*

$$sign(\lambda_1)\epsilon_1, \ldots, sign(\lambda_\alpha)\epsilon_\alpha.$$

In the next three sections we include results that will be used in the proof of Theorem 4.1. Instead of finding an explicit reducing Jordan pair for $L(\lambda)$, the proof of this theorem will be based on Proposition 2.4 as follows:

- We first find a Jordan pair (Z, J) (not necessarily a reducing pair) for $L(\lambda)$ by constructing a Jordan chain for each eigenvalue of $L(\lambda)$ from a Jordan chain of $P(\lambda)$ associated with the same eigenvalue.
- Because (Z, J) is a Jordan pair, we have $J = Z^{-1} C_L Z$. However, since (Z, J) may not be a reducing pair, $(J, Z^* B_L Z)$ may not be a canonical pair for $L(\lambda)$. Thus, we show that there exists a nonsingular matrix Q that commutes with J and such that $Z^* B_L Z = Q^* P_{\epsilon,J} Q$ for some set of signs ϵ. By Proposition 2.4, ϵ is the sign characteristic of $L(\lambda)$.

5. Jordan pairs of linearizations in $\mathbb{DL}(P)$. In this section we construct Jordan chains for pencils in $\mathbb{DL}(P)$ from Jordan chains of $P(\lambda)$ associated with the same eigenvalue, generalizing the construction of eigenvectors of pencils in $\mathbb{DL}(P)$ from eigenvectors of $P(\lambda)$ given in Theorem 3.2.

We start with a technical lemma.

Lemma 5.1. *Let $P(\lambda)$ be a matrix polynomial of degree k, and let $L(\lambda) = D(P,v) \in \mathbb{DL}(P)$, with $v \in \mathbb{C}^k$. Then,*

$$v \otimes P(\lambda) = L(\lambda)\left(\Lambda(\lambda) \otimes I_n\right), \tag{5.1}$$

$$v \otimes P^{(j)}(\lambda) = L(\lambda)\left(\Lambda^{(j)}(\lambda) \otimes I_n\right) + jL'(\lambda)\left(\Lambda^{(j-1)}(\lambda) \otimes I_n\right), \ j > 0. \tag{5.2}$$

(In particular, for $j > k$, $v \otimes P^{(j)}(\lambda) = 0$.) Moreover, for $r \geq 1$,

$$v \otimes \left(\sum_{j=1}^{r} \frac{1}{j!} P^{(j)}(\lambda) \right) = L(\lambda) \sum_{j=1}^{r} \left(\frac{1}{j!} \Lambda^{(j)}(\lambda) \otimes I_n \right) + L'(\lambda) \sum_{j=1}^{r} \left(\frac{1}{(j-1)!} \Lambda^{(j-1)}(\lambda) \otimes I_n \right). \tag{5.3}$$

Proof. Condition (5.1) follows from the definition of $\mathbb{L}_1(P)$ and the fact that $\mathbb{DL}(P) \subseteq \mathbb{L}_1(P)$.

To prove (5.2), we proceed by induction on j. Differentiating (5.1) with respect to λ, we get

$$v \otimes P'(\lambda) = L(\lambda)\left(\Lambda'(\lambda) \otimes I_n\right) + L'(\lambda)\left(\Lambda(\lambda) \otimes I_n\right),$$

and (5.2) holds for $j = 1$. Next, suppose that (5.2) holds for some $j > 0$. Keeping in mind that $L^{(2)}(\lambda) = 0$, since $L(\lambda)$ is a polynomial of degree 1, and using the inductive hypothesis, we have

$$\begin{aligned} v \otimes P^{(j+1)}(\lambda) &= L(\lambda)\left(\Lambda^{(j+1)}(\lambda) \otimes I_n\right) + L'(\lambda)\left(\Lambda^{(j)}(\lambda) \otimes I_n\right) + jL'(\lambda)\left(\Lambda^{(j)}(\lambda) \otimes I_n\right) \\ &= L(\lambda)\left(\Lambda^{(j+1)}(\lambda) \otimes I_n\right) + (j+1)L'(\lambda)\left(\Lambda^{(j)}(\lambda) \otimes I_n\right), \end{aligned}$$

and, thus, (5.2) follows. Note that, if $j \geq k$, then $\Lambda^{(j)}(\lambda) = 0$. Condition (5.3) follows by multiplying (5.2) by $1/j!$ for each j and summing the expressions for $j = 1, \ldots, r$, collecting like terms. □

Theorem 5.2. *Let $P(\lambda)$ be an $n \times n$ regular matrix polynomial of degree k. Let $L(\lambda) \in \mathbb{DL}(P)$ be a linearization of $P(\lambda)$. Let λ_0 be a finite eigenvalue of $P(\lambda)$. Then, $\{x_1, \ldots, x_r\}$ is a Jordan chain of $P(\lambda)$ corresponding to λ_0 if and only if $\{z_1, \ldots, z_r\}$ is a Jordan chain of $L(\lambda)$ corresponding to λ_0, where*

$$z_\ell = \sum_{j=0}^{\ell-1} \left(\frac{1}{j!} \Lambda^{(j)}(\lambda_0) \otimes x_{\ell-j} \right), \quad \ell = 1, \ldots, r. \tag{5.4}$$

Proof. For $r = 1$, the claim follows from Theorem 3.2. Recall that a single vector forms a Jordan chain if and only if it is an eigenvector.

Next, assume that the theorem holds for some $r \geq 1$. We want to show that $\{x_1, \ldots, x_{r+1}\}$ is a Jordan chain for $P(\lambda)$ associated with λ_0 if and only if $\{z_1, \ldots, z_{r+1}\}$ is a Jordan chain for $L(\lambda)$ associated with λ_0. By the definition of Jordan chain (see Section 2) and, using the inductive hypothesis, it is enough to see that

$$\sum_{j=0}^{r} \frac{1}{j!} P^{(j)}(\lambda_0) x_{r+1-j} = 0$$

if and only if

$$\sum_{j=0}^{r} \frac{1}{j!} L^{(j)}(\lambda_0) z_{r+1-j} = L(\lambda_0) z_{r+1} + L'(\lambda_0) z_r = 0.$$

Assume that $L(\lambda) = D(P, v)$, for some $v \in \mathbb{C}^k$. We have

$$\begin{aligned}
&L(\lambda_0)z_{r+1} + L'(\lambda_0)z_r \\
&= L(\lambda_0)\sum_{j=0}^{r}\left(\frac{1}{j!}\Lambda^{(j)}(\lambda_0)\otimes x_{r+1-j}\right) + L'(\lambda_0)\sum_{j=1}^{r}\left(\frac{1}{(j-1)!}\Lambda^{(j-1)}(\lambda_0)\otimes x_{r+1-j}\right) \\
&= L(\lambda_0)(\Lambda(\lambda_0)\otimes x_{r+1}) + \sum_{j=1}^{r}\left[L(\lambda_0)\left(\frac{1}{j!}\Lambda^{(j)}(\lambda_0)\otimes x_{r+1-j}\right) + \right.\\
&\left. L'(\lambda_0)\left(\frac{1}{(j-1)!}\Lambda^{(j-1)}(\lambda_0)\otimes x_{r+1-j}\right)\right] \\
&= (v\otimes P(\lambda_0))\, x_{r+1} + \sum_{j=1}^{r}\left(v\otimes \frac{1}{j!}P^{(j)}(\lambda_0)\right) x_{r+1-j} = v\otimes \sum_{j=0}^{r}\frac{1}{j!}P^{(j)}(\lambda_0)x_{r+1-j},
\end{aligned}$$

where the third equality follows from Lemma 5.1. Since $L(\lambda)$ is a linearization of a regular $P(\lambda)$, the vector v is nonzero. Thus, the desired equivalence follows. ◻

Note that the Jordan chain $\{z_1, \ldots, z_r\}$ for $D(P, v)$ constructed from a Jordan chain $\{x_1, \ldots, x_r\}$ using (5.4) does not depend on the vector v.

Definition 5.3. *We say that the Jordan chain $\{z_1, \ldots, z_r\}$ for $D(P, v)$ constructed from a Jordan chain $\{x_1, \ldots, x_r\}$ for $P(\lambda)$ using (5.4) is the* Jordan chain for $D(P, v)$ associated with $\{x_1, \ldots, x_r\}$.

Definition 5.4. *Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree k with nonsingular leading coefficient and let (X, J) be a Jordan pair for $P(\lambda)$. Let Z be the matrix obtained from X by replacing each maximal Jordan chain $\{x_1, \ldots, x_r\}$ in X, associated with some eigenvalue λ_0 of $P(\lambda)$, by $\{z_1, \ldots, z_r\}$, where z_l, $l = 1, \ldots, r$ is as in (5.4). We call Z the extended matrix associated with X and denote it by $Z(X)$.*

Next we construct a Jordan pair for $D(P, v)$ from a Jordan pair (X, J) of $P(\lambda)$.

In what follows, we denote

$$R := \begin{bmatrix} 0 & 0 & \cdots & 0 & I_n \\ 0 & 0 & \cdots & I_n & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & I_n & \cdots & 0 & 0 \\ I_n & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{C}^{nk\times nk}. \tag{5.5}$$

Proposition 5.5. *Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree k with nonsingular leading coefficient and let (X, J) be a Jordan pair for $P(\lambda)$. Let $Z(X)$ be the extended matrix associated with X. Then, $Z(X) = RS$, where S is the (X, J)-matrix and R is as in (5.5). Moreover, if $D(P, v) \in \mathbb{DL}(P)$ is a linearization of $P(\lambda)$, then $(Z(X), J)$ is a Jordan pair for $D(P, v)$.*

Proof. Suppose that $J = J_{l_1}(\lambda_1) \oplus \cdots \oplus J_{l_s}(\lambda_s)$ and let $Z(X) = [Z_{ij}]_{\substack{i=1,\ldots,k\\ j=1,\ldots,s}}$ with Z_{ij} of size $n \times l_j$ and $X = [X_j]_{j=1,\ldots,s}$ with X_j of size $n \times l_j$.

Taking into account that S is as in (2.3), it can be easily seen that the equality $Z(X) = RS$ is equivalent to

$$Z_{kj} = X_j, \quad Z_{ij} = Z_{i+1,j}J_{l_j}(\lambda_j), \text{ for } i = 1, \ldots, k-1,\ j = 1, \ldots, s.$$

Fix $i \in \{1, \ldots, k-1\}$ and $j \in \{1, \ldots, s\}$. Let

$$X_j = [x_1^{(j)}, \ldots, x_{l_j}^{(j)}] \text{ and } Z_{ij} = [z_1^{(ij)}, \ldots, z_{l_j}^{(ij)}],$$

where $x_l^{(j)}$ and $z_l^{(ij)}$ are the lth columns of X_j and Z_{ij}, respectively. Taking into account the definition of $Z(X)$, we have that, for $l = 1, \ldots, l_j$, $z_l^{(ij)}$ is the ith block row of

$$\sum_{w=0}^{l-1} \left(\frac{1}{w!} \Lambda^{(w)}(\lambda_j) \otimes x_{l-w} \right).$$

A calculation shows that the ith block-row of $\Lambda^{(w)}(\lambda)$ is $(k-i)! q_{k-i-w}(\lambda)$, where $q_t(\lambda) = \frac{1}{t!}\lambda^t$ if $t \geq 0$ and $q_t = 0$ otherwise. Thus, we have

$$z_l^{(ij)} = \sum_{w=0}^{l-1} \binom{k-i}{w} \lambda_j^{k-i-w} x_{l-w}^{(j)}.$$

(For nonnegative integers a, b, we assume $\binom{a}{b}\lambda^{a-b} = 0$ if $b > a$. We also assume $0^0 = 1$). We have $z_l^{(kj)} = x_l^{(j)}$, $l = 1, \ldots, l_j$, implying $Z_{kj} = X_j$. Now suppose that $i < k$. We have $z_1^{(ij)} = \lambda_j^{k-i} x_l^{(j)} = \lambda_j z_1^{(i+1,j)}$. Also, for $l = 1, \ldots, l_j - 1$,

$$\begin{aligned}
&\lambda_j z_{l+1}^{(i+1,j)} + z_l^{(i+1,j)} \\
&= \sum_{w=0}^{l} \binom{k-i-1}{w} \lambda_j^{k-i-w} x_{l+1-w}^{(j)} + \sum_{w=0}^{l-1} \binom{k-i-1}{w} \lambda_j^{k-i-1-w} x_{l-w}^{(j)} \\
&= \lambda_j^{k-i} x_{l+1}^{(j)} + \sum_{w=1}^{l} \left[\binom{k-i-1}{w} + \binom{k-i-1}{w-1} \right] \lambda_j^{k-i-w} x_{l+1-w}^{(j)} \\
&= \lambda_j^{k-i} x_{l+1}^{(j)} + \sum_{w=1}^{l} \binom{k-i}{w} \lambda_j^{k-i-w} x_{l+1-w}^{(j)} = z_{l+1}^{(i,j)}.
\end{aligned}$$

Thus, it follows that $Z_{ij} = Z_{i+1,j} J_{l_j}(\lambda_j)$.

Now we show the second claim. Since the columns of X form Jordan chains for $P(\lambda)$, by definition of $Z(X)$ and Theorem 5.2, the corresponding columns of $Z(X)$ form Jordan chains for $L(\lambda) := D(P, v) = \lambda L_1 - L_0$, and thus, for $C_L = L_1^{-1} L_0$. Therefore, $C_L Z(X) = Z(X) J$. Since we have proven that $Z(X) = RS$, and R and S are invertible, it follows that $Z(X)$ is also invertible. Thus, we have $J = Z(X)^{-1} C_L Z(X)$, which implies that $(Z(X), J)$ is a Jordan pair for $D(P, v)$. ∎

The following result is a nice consequence of Proposition 5.5. It gives an interesting relation between the companion matrix of $P(\lambda)$ and the coefficients of any linearization in $\mathbb{DL}(P)$. It also emphasizes the fact that the Jordan chains for a pencil in $\mathbb{DL}(P)$ do not depend on the ansatz vector associated with the pencil.

Corollary 5.6. *Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree k with nonsingular leading coefficient. If $L(\lambda) = \lambda L_1 - L_0 \in \mathbb{DL}(P)$ is a linearization of $P(\lambda)$, then $L_1^{-1} L_0 = R C_P R$, where R is as in (5.5).*

Proof. If (X, J) is a Jordan pair for $P(\lambda)$ and S is the (X, J)-matrix, we have $J = S^{-1} C_P S$. From Proposition 5.5, $Z(X) = RS$ and $J = Z(X)^{-1} C_L Z(X)$, where $Z(X)$ is the extended matrix associated with X. Since $C_L = L_1^{-1} L_0$, the claim follows. ∎

As follows from Proposition 5.5, if $D(P, v) \in \mathbb{H}(P)$ is a linearization of a Hermitian matrix polynomial $P(\lambda)$ with nonsingular leading coefficient, then $(Z(X), J)$ is a Jordan pair for $D(P, v)$, that is,

$$J = Z(X)^{-1} C_{D(P,v)} Z(X).$$

Unfortunately, in general, the pair $(Z(X), J)$ is not a reducing Jordan pair associated with $L(\lambda)$ (see Definition 2.3), even if (X, J) is a reducing pair for $P(\lambda)$, because $Z(X)^* B_L Z(X)$ may not be of the form $P_{\epsilon,J}$ for a set of signs ϵ. However, the matrix

$$H(X, J, v) := Z(X)^* B_{D(P,v)} Z(X) \tag{5.6}$$

will play an important role in finding the sign characteristic of $D(P, v)$. In the next section we study the block-structure of $H(X, J, v)$.

6. Block-Hankel structure of $H(X, J, v)$. Let $P(\lambda)$ be a Hermitian matrix polynomial with nonsingular leading coefficient, and $D(P, v) \in \mathbb{H}(P)$ be a linearization of $P(\lambda)$. If (X, J) is a Jordan pair for $P(\lambda)$, we show here that the leading principal submatrix, say H_1, of the matrix $H(X, J, v)$ (given in (5.6)) with the same number of rows and columns as the submatrix of J formed by the blocks corresponding to the real eigenvalues of $P(\lambda)$, has a certain "block-Hankel" structure. This matrix H_1 will play a crucial role in our analysis of the sign characteristic of $D(P, v)$. We will see later (Lemma 8.2 applied to $D(P, v)$) that, in fact, $H(X, J, v) = H_1 \oplus H_2$, for some matrix H_2.

We start with an auxiliary lemma, which generalizes Lemma 2.8 in [1], where it was proven that, if λ_0 is a finite eigenvalue of $P(\lambda)$ (and, therefore of a linearization $L(\lambda) \in \mathbb{H}(P)$) and x is a (right) eigenvector of $P(\lambda)$ associated with λ_0, then

$$z^* L'(\lambda_0) z = p(\lambda_0; v)\,(x^* P'(\lambda_0) x)\,,$$

where $z = \Lambda(\lambda_0) \otimes x$ is a right eigenvector of $L(\lambda)$ and $p(\lambda; v)$ is defined in (3.2).

Given two complex vectors x, y of the same size, we denote, as usual, $\langle x, y\rangle := y^* x$.

LEMMA 6.1. *Let $P(\lambda)$ be an $n \times n$ Hermitian matrix polynomial of degree k with nonsingular leading coefficient and let $L(\lambda) = D(P, v) = \lambda L_1 - L_0 \in \mathbb{H}(P)$ be a linearization of $P(\lambda)$. Let (X, J) be a Jordan pair for $P(\lambda)$. Let λ_0 be a real eigenvalue of $P(\lambda)$ and y_1 be an associated (right) eigenvector of $P(\lambda)$ in X. Let $\{x_1, \ldots, x_s\}$ be a maximal Jordan chain of $P(\lambda)$ corresponding to λ_0 obtained from X and let $\{z_1, \ldots, z_s\}$ be the associated Jordan chain for $L(\lambda)$ (as in Theorem 5.2). Then, for $1 \le r \le s$,*

$$\langle \Lambda(\lambda_0) \otimes y_1, L_1 z_r\rangle = p(\lambda_0; v) \left\langle y_1, \sum_{j=1}^{r} \frac{1}{j!} P^{(j)}(\lambda_0) x_{r+1-j} \right\rangle, \tag{6.1}$$

where $p(\lambda; v)$ is the v-polynomial. Moreover, if (X, J) is a reducing Jordan pair associated with $P(\lambda)$, then

$$\langle \Lambda(\lambda_0) \otimes y_1, L_1 z_r\rangle = \begin{cases} \epsilon_0 p(\lambda_0; v), & \text{if } r = s \text{ and } y_1 = x_1, \\ 0, & \text{if } r < s \text{ or } y_1 \neq x_1, \end{cases}$$

where ϵ_0 is the sign in the sign characteristic of $P(\lambda)$ associated with the Jordan chain $\{x_1, \ldots, x_s\}$.

Proof. Denote $w_1 := \Lambda(\lambda_0) \otimes y_1$. Taking into account (5.4) and the fact that $L'(\lambda_0) = L_1$, we get

$$\langle w_1, L_1 z_r\rangle = \left\langle w_1, L'(\lambda_0) \sum_{j=0}^{r-1} \left(\frac{1}{j!} \Lambda^{(j)}(\lambda_0) \otimes x_{r-j} \right) \right\rangle$$

Then, using Lemma 5.1, we have

$$\begin{aligned}
&\langle w_1, L_1 z_r\rangle \\
&= \left\langle w_1, v\otimes\left(\sum_{j=1}^{r}\frac{1}{j!}P^{(j)}(\lambda_0)x_{r+1-j}\right) - L(\lambda_0)\left(\sum_{j=1}^{r}\frac{1}{j!}\Lambda^{(j)}(\lambda_0)\otimes x_{r+1-j}\right)\right\rangle \\
&= \left\langle w_1, v\otimes\left(\sum_{j=1}^{r}\frac{1}{j!}P^{(j)}(\lambda_0)x_{r+1-j}\right)\right\rangle - \left\langle L(\lambda_0)w_1, \sum_{j=1}^{r}\frac{1}{j!}\Lambda^{(j)}(\lambda_0)\otimes x_{r+1-j}\right\rangle \\
&= \left\langle w_1, v\otimes\left(\sum_{j=1}^{r}\frac{1}{j!}P^{(j)}(\lambda_0)x_{r+1-j}\right)\right\rangle = \langle\Lambda_0, v\rangle\langle y_1, \sum_{j=1}^{r}\frac{1}{j!}P^{(j)}(\lambda_0)x_{r+1-j}\rangle
\end{aligned}$$

where the second equality follows because $L(\lambda_0)$ is Hermitian and the third equality follows because w_1 is an eigenvector for $L(\lambda)$ associated with λ_0. The last equality follows from the definition of w_1, the definition of the inner product, and the following property of the Kronecker product: given column vectors u_1, u_2, u_3, u_4 of appropriate sizes, we have $(u_1\otimes u_2)^*(u_3\otimes u_4) = (u_1^*u_3)(u_2^*u_4)$. Now (6.1) follows taking into account that $p(\lambda_0; v) = \langle\Lambda(\lambda_0), v\rangle$. The second claim follows from (6.1) and Theorem 1.11 in [7]. □

Let (X, J) be a Jordan pair for $P(\lambda)$ and let $L(\lambda) = D(P, v) = \lambda L_1 - L_0 \in \mathbb{H}(P)$. Let X_i denote a Jordan chain of $P(\lambda)$ in X, with sign characteristic ϵ_i, and let Z_i be the corresponding Jordan chain in $Z(X)$. Then, Lemma 6.1 implies that, in the ith main diagonal block of $H(X, J, v)$ (that is, the block $Z_i^* L_1 Z_i$), the element in the first column and last row is $\epsilon_i p(\lambda_0; v)$, while the rest of the elements in the first column are zero. Additionally, in the non-diagonal blocks, corresponding to products of two distinct Jordan chains associated with the same real eigenvalue, the first column is all zeros.

We now give a more detailed description of the structure of $H(X, J, v)$. We will use the following notation and concepts.

We recall that a $p\times q$ matrix $B = [b_{ij}]$ is said to be *Hankel* if $b_{i,j-1} = b_{i-1,j}$, for all $i = 2,\dots,p$ and $j = 2,\dots,q$. When B is a $p\times p$ matrix, we call the ordered set of entries $b_{i,p+1-i}$, $i = 1,\dots,p$ the *main skew-diagonal* of B.

We denote by $H_{p\times q}$ the set of $p\times q$ Hankel matrices A of the form

$$A = \begin{bmatrix} 0 & B \end{bmatrix}, \text{ if } p\le q, \quad \text{and } A = \begin{bmatrix} 0 \\ B \end{bmatrix}, \text{ if } p\ge q,$$

where B is a square Hankel matrix with zeros above the main skew-diagonal. Note that, for $A = [a_{ij}] \in H_{p\times q}$, we have $a_{ij} = 0$ for $i + j < p + 1 + \max\{0, q-p\}$. We denote by $H^0_{p\times q}$ the subset of matrices in $H_{p\times q}$ in which B has zeros on the main skew-diagonal.

DEFINITION 6.2. *Let $A = [A_{ij}]_{i,j=1,\dots,l}$, with $A_{ij}\in\mathbb{C}^{s_i\times s_j}$. We say that A is an* $(s_1, s_2, \cdots, s_l)$ block-Hankel matrix *if $A_{ii}\in H_{s_i\times s_i}$ and $A_{ij}\in H^0_{s_i\times s_j}$ for $i\ne j$, and we call the entries $\alpha_1,\dots,\alpha_l$ on the main skew-diagonal of the blocks A_{ii}, $i = 1,\dots,l$, the* main skew-diagonal entries of A.

LEMMA 6.3. *Let $P(\lambda)$ be an $n\times n$ Hermitian matrix polynomial of degree k with nonsingular leading coefficient and let (X, J) be a reducing Jordan pair for $P(\lambda)$. Let λ_0 be a real eigenvalue of $P(\lambda)$. Let $L(\lambda) = D(P, v)\in\mathbb{H}(P)$ be a linearization of $P(\lambda)$. Let $\{x_1,\dots,x_c\}$ and $\{y_1,\dots,y_r\}$ be maximal Jordan chains of $P(\lambda)$ associated with*

λ_0 *in* X*, and let* $\{z_1,\ldots,z_c\}$ *and* $\{w_1,\ldots,w_r\}$ *be, respectively, the associated Jordan chains for* $L(\lambda)$*. Let* $Z := [z_1,\ldots,z_c]$ *and* $W := [w_1,\ldots,w_r]$*. Then the following conditions hold.*

i) $Z^*B_LZ \in H_{c\times c}$ *has real entries and has main skew-diagonal entries equal to* $\epsilon_0 p(\lambda_0;v)$*, where* ϵ_0 *is the sign in the sign characteristic of* $P(\lambda)$ *corresponding to the Jordan chain* $\{x_1,\ldots,x_c\}$*.*

ii) $Z^*B_LW \in H^0_{c\times r}$ *if* Z *and* W *are distinct.*

Proof. Since, by Theorem 5.2, the columns of Z form a Jordan chain for $L(\lambda)$ at λ_0, we have

$$\sum_{j=0}^{i-1}\frac{1}{j!}L^{(j)}(\lambda_0)z_{i-j}=0$$

for $i\le c$. This is equivalent to

$$L(\lambda_0)z_1=0 \quad\text{and}\quad L(\lambda_0)z_i+L'(\lambda_0)z_{i-1}=0 \tag{6.2}$$

for $i=2,\ldots,c$. We prove that the $c\times c$ matrix $Z^*B_LZ=[\langle B_Lz_j,z_i\rangle]_{i,j}$ is a Hankel matrix by showing that

$$\langle B_Lz_{j-1},z_i\rangle=\langle B_Lz_j,z_{i-1}\rangle,\quad i,j=2,\ldots,c.$$

Taking into account (6.2) and the fact that both $B_L=L'(\lambda_0)$ and $L(\lambda_0)$ are Hermitian, we have

$$\begin{aligned}\langle L'(\lambda_0)z_{j-1},z_i\rangle &= \langle -L(\lambda_0)z_j,z_i\rangle\\ &= \langle -z_j,L(\lambda_0)z_i\rangle=\langle -z_j,-L'(\lambda_0)z_{i-1}\rangle\\ &= \langle z_j,L'(\lambda_0)z_{i-1}\rangle=\langle L'(\lambda_0)z_j,z_{i-1}\rangle,\end{aligned}$$

showing the claim. Moreover, since $H(X,J,v)$ is Hermitian, the entries of the Hankel matrix Z^*B_LZ are real.

A similar argument shows that the $c\times r$ matrix $Z^*B_LW=[\langle B_Lz_j,w_i\rangle]_{i,j}$ also has constant elements along the "skew-diagonals", that is, $\langle B_Lz_{j-1},w_i\rangle=\langle B_Lz_j,w_{i-1}\rangle$, for $i=2,\ldots,c$ and $j=2,\ldots,r$.

Let $\{z_1,\ldots z_c\}$ and $\{w_1,\ldots,w_r\}$ be the columns of the (X,J)-matrix corresponding to $\{x_1,\ldots,x_c\}$ and $\{y_1,\ldots,y_r\}$, respectively. To complete our proof, it is enough to show that

$$\langle B_Lz_1,z_j\rangle=0,\quad\text{if}\quad j=1,\ldots,c-1, \tag{6.3}$$

$$\langle B_Lz_1,z_c\rangle=\epsilon_0 p(\lambda_0;v), \tag{6.4}$$

and, if $Z\ne W$,

$$\langle B_Lz_1,w_j\rangle=0,\quad\text{if}\quad j=1,\ldots,c, \tag{6.5}$$

$$\langle B_Lz_j,w_1\rangle=0\quad\text{if}\quad j=1,\ldots,r. \tag{6.6}$$

Conditions (6.3), (6.4), (6.5) and (6.6) follow from Lemma 6.1. Note that to obtain (6.6) we use the fact that

$$\langle B_Lz_j,w_1\rangle=\langle w_1,B_Lz_j\rangle^*.$$

□

The next corollary is an immediate consequence of the previous lemma.

COROLLARY 6.4. *Let $P(\lambda)$ be an $n \times n$ Hermitian matrix polynomial of degree k with nonsingular leading coefficient and let $(J, P_{\epsilon,J})$ be a canonical pair for $P(\lambda)$ with reducing Jordan pair (X, J). Suppose that $J = J_1 \oplus J_2$, where J_2 has nonreal eigenvalues and $J_1 = J_{11} \oplus \cdots \oplus J_{1\alpha}$ has real eigenvalues, with*

$$J_{1i} = J_{s_{i,1}}(\lambda_i) \oplus \cdots \oplus J_{s_{i,l_i}}(\lambda_i),$$

and $\lambda_i \neq \lambda_j$ for $i \neq j$. Let the main diagonal block of $P_{\epsilon,J}$ corresponding to J_{1i} be $\epsilon_{i,1}\mathcal{R}_{s_{i,1}} \oplus \cdots \oplus \epsilon_{i,l_i}\mathcal{R}_{s_{i,l_i}}$. Let $L(\lambda) = D(P, v) \in \mathbb{H}(P)$ be a linearization of $P(\lambda)$ and let $Z = Z(X)$ be the extended matrix associated with X, where

$$Z = \begin{bmatrix} Z_{11} & \cdots & Z_{1l_i} & Z_2 \end{bmatrix},$$

and the number of columns of Z_{1i} and J_{1i} is the same. Then $Z_{1i}^* B_L Z_{1i}$, $i = 1, \ldots, \alpha$, is an $(s_{i,1}, s_{i,2}, \cdots, s_{i,l_i})$ block-Hankel matrix with real main skew-diagonal entries $\epsilon_{i,1}p(\lambda_i; v)$, $\ldots$,$\epsilon_{i,l_i}p(\lambda_i; v)$.

Given a reducing pair (X, J) of a regular Hermitian $P(\lambda)$, by Proposition 5.5, $(Z(X), J)$, where $Z(X)$ is the extended matrix associated with X, is a Jordan pair for a linearization $L(\lambda) \in \mathbb{H}(P)$. By Proposition 2.4 applied to $L(\lambda)$ (note that $Z(X)$ coincides with the $(Z(X), J)$-matrix), a pair $(J, P_{\epsilon',J})$ is a canonical pair for $L(\lambda)$ if and only if there exists a nonsingular matrix Q that commutes with J and such that

$$Q^* P_{\epsilon',J} Q = Z(X)^* B_L Z(X) = H(X, J, v).$$

It is well known that matrices that commute with a Jordan matrix are special cases of block-Toeplitz matrices. In the next section we study the existence of block-Toeplitz solutions Q of a general equation of the form $Q^*\mathcal{L}Q = H$, where H has a block-Hankel structure and $\mathcal{L}$ is a direct sum of matrices as in (5.5), multiplied by some constants ± 1.

7. On the block-Toeplitz solutions of the equation $Q^*\mathcal{L}Q = H$ for a block-Hankel H. The main result in this section is Theorem 7.8, which will be used in Section 8 to prove our main theorem (Theorem 4.1) and has also independent interest.

We start with some definitions.

We recall that a $p \times q$ matrix $B = [b_{ij}]$ is said to be *Toeplitz* if $b_{ij} = b_{i+1,j+1}$, $i = 1, \ldots, p-1$ and $j = 1, \ldots, q-1$.

In what follows we denote by $T_{p\times q}$ the set of $p \times q$ Toeplitz matrices A of the form

$$A = \begin{bmatrix} 0 & B \end{bmatrix}, \text{ if } p \leq q, \quad \text{ and } A = \begin{bmatrix} B \\ 0 \end{bmatrix}, \text{ if } p \geq q,$$

where B is a square upper triangular Toeplitz matrix. Note that, for $A = [a_{ij}] \in T_{p\times q}$, $a_{ij} = 0$ for $i > j - \max\{0, q-p\}$. We denote by $T^0_{p\times q}$ the subset of matrices in $T_{p\times q}$ such that B is nilpotent, that is, its main diagonal entries are all zero. Note that $A \in T_{p\times q}$ (resp. $A \in T^0_{p\times q}$) if and only if $\mathcal{R}_p A \in H_{p\times q}$ (resp. $\mathcal{R}_p A \in H^0_{p\times q}$), where $\mathcal{R}_p$ is defined in (2.4).

DEFINITION 7.1. *Let $A = [A_{ij}]_{i,j=1,\ldots,l}$, with $A_{ij} \in \mathbb{C}^{s_i \times s_j}$. We say that A is an $(s_1, s_2, \ldots, s_l)$ block-Toeplitz matrix if $A_{ii} \in T_{s_i\times s_i}$ and $A_{ij} \in T^0_{s_i\times s_j}$ for $i \neq j$.*

The following lemma can be easily verified.

LEMMA 7.2. *Let* $\mathbf{a} = [a_1, \ldots, a_m]^T \in \mathbb{C}^m$. *Then*

$$\mathbf{a}^T \mathcal{R}_m \mathbf{a} = \begin{cases} a_1^2, & m = 1; \\ \sum_{l=1}^{\frac{m}{2}} 2a_l a_{m+1-l}, & \text{if } m > 1 \text{ and } m \text{ is even;} \\ a_{\frac{m+1}{2}}^2 + \sum_{l=1}^{\frac{m-1}{2}} 2a_l a_{m+1-l}, & \text{if } m > 1 \text{ and } m \text{ is odd.} \end{cases}$$

PROPOSITION 7.3. *Let* $B \in H_{p\times p}$ *be a real matrix with all the entries on the main skew-diagonal equal to* $b_1 \neq 0$. *Then, there exists a real nonsingular matrix* $A \in T_{p\times p}$ *such that* $A^T \mathcal{R}_p A = sign(b_1)B$.

Proof. Let B be the real Hankel matrix given by

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 & b_1 \\ 0 & & & b_1 & b_2 \\ \vdots & & \iddots & \iddots & \vdots \\ 0 & b_1 & b_2 & & b_{p-1} \\ b_1 & b_2 & \cdots & b_{p-1} & b_p \end{bmatrix} \in H_{p\times p},$$

with $b_1 \neq 0$. Let $\gamma = sign(b_1)$. Consider the $p \times p$ Toeplitz matrix

$$Q(B) := \begin{bmatrix} q_1 & q_2 & \cdots & q_{p-1} & q_p \\ & \ddots & \ddots & \vdots & q_{p-1} \\ & & \ddots & q_2 & \vdots \\ & & & q_1 & q_2 \\ 0 & & & & q_1 \end{bmatrix}$$

defined recursively by

$$q_1 = \sqrt{\gamma b_1},$$

$$q_m = \frac{1}{2q_1}\left(\gamma b_m - \sum_{\ell=2}^{\frac{m}{2}} 2q_\ell q_{m-\ell+1}\right), \quad \text{if } m > 1 \text{ and } m \text{ is even,} \tag{7.1}$$

$$q_m = \frac{1}{2q_1}\left(\gamma b_m - q_{\frac{m+1}{2}}^2 - \sum_{\ell=2}^{\frac{m-1}{2}} 2q_\ell q_{m-\ell+1}\right), \quad \text{if } m > 1 \text{ and } m \text{ is odd.}$$

Notice that $Q(B)$ is nonsingular since $q_1 \neq 0$, and is real since B is. Now we compute the matrix $Q(B)^T \mathcal{R}_p Q(B)$. Let $Q(B)^T := [a_{ij}]$ and $\mathcal{R}_p Q(B) = [b_{ij}]$. For $k = 1, \ldots, p$, we have

$$a_{ik} = \begin{cases} q_{i-k+1} & \text{if } k \leq i \\ 0 & \text{if } k > i \end{cases} \qquad \text{and} \qquad b_{kj} = \begin{cases} q_{k+j-p} & \text{if } k+j \geq p+1 \\ 0 & \text{if } k+j < p+1. \end{cases}$$

Suppose that $i+j \leq p$. Then, for $k = 1, \ldots, p$, either $a_{ik} = 0$ or $b_{kj} = 0$. Thus, when $i+j \leq p$, the (i,j)th entry of $Q(B)^T \mathcal{R}_p Q(B)$ is 0. Now suppose that $i+j > p$. Notice that this implies that $p+1-j \leq i$. Then, the (i,j)th entry of $Q(B)^T \mathcal{R}_p Q(B)$ is

$$\begin{aligned}\sum_{k=1}^{p} a_{ik} b_{kj} &= a_{i,p+1-j} b_{p+1-j,j} + \cdots + a_{ii} b_{ij} \\ &= q_{i+j-p} q_1 + \cdots + q_1 q_{i+j-p}.\end{aligned}$$

Thus, the matrix $Q(B)^T \mathcal{R}_p Q(B)$ is given by

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & \mathbf{q}_1^T \mathcal{R}_1 \mathbf{q}_1 \\ 0 & 0 & \cdots & \mathbf{q}_1^T \mathcal{R}_1 \mathbf{q}_1 & \mathbf{q}_2^T \mathcal{R}_2 \mathbf{q}_2 \\ \vdots & \vdots & ⋰ & ⋰ & \vdots \\ 0 & \mathbf{q}_1^T \mathcal{R}_1 \mathbf{q}_1 & \cdots & \mathbf{q}_{p-2}^T \mathcal{R}_{p-2} \mathbf{q}_{p-2} & \mathbf{q}_{p-1}^T \mathcal{R}_{p-1} \mathbf{q}_{p-1} \\ \mathbf{q}_1^T \mathcal{R}_1 \mathbf{q}_1 & \mathbf{q}_2^T \mathcal{R}_2 \mathbf{q}_2 & \cdots & \mathbf{q}_{p-1}^T \mathcal{R}_{p-1} \mathbf{q}_{p-1} & \mathbf{q}_p^T \mathcal{R}_p \mathbf{q}_p \end{bmatrix}, \tag{7.2}$$

where $\mathbf{q}_i = [q_1, \ldots, q_i]^T$ for $i = 1, \ldots, p$. By Lemma 7.2, $\mathbf{q}_1^T R_1 \mathbf{q}_1 = q_1^2 = sign(b_1) b_1$. Also, for $m > 1$ even, we have

$$\begin{aligned}\mathbf{q}_m^T \mathcal{R}_m \mathbf{q}_m &= \sum_{l=1}^{m/2} 2 q_l q_{m+1-l} \\ &= 2 q_1 q_m + \sum_{l=2}^{m/2} 2 q_l q_{m+1-l} = sign(b_1) b_m,\end{aligned}$$

where the first equality follows from Lemma 7.2 and the third equality follows from (7.1). The proof for $m > 1$ odd is similar. Thus, the claim follows with $A = Q(B)$. □

Next we include some technical results that will be useful to prove Theorem 7.8, which is the main result of this section.

Proposition 7.4. *Let $B \in H_{p\times s}$ and $Q \in T^0_{s\times r}$. Then $BQ \in H^0_{p\times r}$.*

Proof. Let $B := [b_{ij}]$ and $Q := [q_{ij}]$. Note that B is a Hankel matrix, which implies $b_{i,j-1} = b_{i-1,j}$ for $i = 2, \ldots, p$ and $j = 2, \ldots, s$. Also, since Q is a Toeplitz matrix, we have $q_{i,j} = q_{i+1,j+1}$ for $i = 1, \ldots, s-1$ and $j = 1, \ldots, r-1$. Moreover, $b_{ij} = 0$ for $i + j < p + 1 + \max\{0, s-p\}$, and $q_{ij} = 0$ for $i \geq j - \max\{0, r-s\}$. In particular, $b_{i1} = 0$ for $i = 1, \ldots, p-1$, and $q_{sj} = 0$ for $j = 1, \ldots, r$. Let $BQ := [d_{ij}]$. First we show that BQ is Hankel, that is, $d_{i,j-1} = d_{i-1,j}$ for $i = 2, \ldots, p$ and $j = 2, \ldots, r$.

For $i = 2, \ldots, p$ and $j = 2, \ldots, r$, we have

$$\begin{aligned}d_{i,j-1} &= \sum_{k=1}^{s} b_{ik} q_{k,j-1} = \sum_{k=1}^{s-1} b_{ik} q_{k,j-1} = \sum_{k=1}^{s-1} b_{i-1,k+1} q_{k+1,j} \\ &= \sum_{k=2}^{s} b_{i-1,k} q_{kj} = \sum_{k=1}^{s} b_{i-1,k} q_{k,j} = d_{i-1,j}.\end{aligned}$$

Now we just need to see that the first row and the first column of BQ are zero. For $i = 1, \ldots, p$, we have

$$d_{i1} = \sum_{k=1}^{s} b_{ik} q_{k1} = 0,$$

as, for each $k = 1, \ldots, s$, we have $q_{k1} = 0$ by definition of $T^0_{s\times r}$. Similarly, for $j = 1, \ldots, r$, we have

$$d_{1j} = \sum_{k=1}^{s} b_{1k} q_{kj} = 0,$$

as, for each $k = 1, \ldots, s-1$, we have $b_{1k} = 0$ by definition of $H_{p\times q}$, and $q_{sj} = 0$ for $j = 1, \ldots, r$, by definition of $T^0_{s\times r}$. □

PROPOSITION 7.5. *Let $B \in H_{p\times p}$ be nonsingular and $C \in H^0_{p\times r}$. Then $B^{-1}C \in T^0_{p\times r}$.*

Proof. We have $B = \mathcal{R}_p A$, for some nonsingular $A \in T_{p\times p}$, where $\mathcal{R}_p$ is as in (2.4). It is well known that the inverse of a nonsingular upper triangular Toeplitz matrix is still an upper triangular Toeplitz matrix [11, Section 0.9.7]. Thus, $A^{-1} \in T_{p\times p}$ and $\mathcal{R}_p A^{-1} \in H_{p\times p}$. On the other hand, $\mathcal{R}_p C \in T^0_{p\times r}$. By Proposition 7.4, $(\mathcal{R}_p A^{-1})(\mathcal{R}_p C) \in H^0_{p\times r}$. Thus, $B^{-1}C = A^{-1}(\mathcal{R}_p C) \in T^0_{p\times r}$. □

PROPOSITION 7.6. *Let $\mathcal{R}_p$ be as in (2.4). If $A \in T^0_{p\times q}$, then $A^*\mathcal{R}_p A \in H^0_{q\times q}$ and is a real matrix.*

Proof. It is easy to see that $A^*\mathcal{R}_p \in H^0_{q\times p}$. Thus, from Proposition 7.4, it follows that $A^*\mathcal{R}_p A \in H^0_{q\times q}$. Since $A^*\mathcal{R}_p A$ is Hankel and Hermitian, it is real. □

The next result is stated using a notation that allows an immediate application in the proof of Theorem 7.8.

LEMMA 7.7. *Let Q_{11} be a nonsingular $(s_1, s_2, \cdots, s_{l-1})$ block-Toeplitz matrix which is also an upper-triangular matrix, and let*

$$H_{12} = \begin{bmatrix} H_1 \\ \vdots \\ H_{l-1} \end{bmatrix},$$

where $H_i \in H^0_{s_i\times s_l}$. Let

$$R_{11} = t_1\mathcal{R}_{s_1} \oplus \cdots \oplus t_{l-1}\mathcal{R}_{s_{l-1}}, \tag{7.3}$$

where $t_i = \pm 1$ for $i = 1, \ldots, l-1$. Then $Q_{12} = R_{11}Q_{11}^{-*}H_{12}$ can be partitioned as

$$Q_{12} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_{l-1} \end{bmatrix}, \tag{7.4}$$

with $Q_i \in T^0_{s_i\times s_l}$.

Proof. We show that the unique solution of the equation $Q^*_{11}R_{11}X = H_{12}$, say Q_{12}, is of the claimed form.

Since the matrix $Q^*_{11}R_{11}$ is a nonsingular block lower triangular matrix, we can solve the equation $Q^*_{11}R_{11}X = H_{12}$ recursively by forward substitution, obtaining matrices $Q_1, \ldots, Q_{l-1}$ so that Q_{12} as in (7.4) is a solution of the equation. Since $Q^*_{11}R_{11}$ is a nonsingular $(s_1, s_2, \cdots, s_{l-1})$ block-Hankel matrix, and $H_1 \in H^0_{s_1\times s_l}$, taking into account Proposition 7.5, $Q_1 \in T^0_{s_1\times s_l}$. Using Proposition 7.4, in the second step of the recursion to solve $Q^*_{11}R_{11}X = H_{12}$, we get an equation of the form $B_2Q_2 = C_2$, with $B_2 \in H_{s_2\times s_2}$ nonsingular and $C_2 \in H^0_{s_2\times s_l}$, and applying Proposition 7.5 again, we get that Q_2 has the desired form. In general, in step i we have an equation of the form $B_iQ_i = C_i$, with $B_i \in H_{s_i\times s_i}$ nonsingular, and $C_i \in H^0_{s_i\times s_l}$ and the result follows from Proposition 7.4. □

We now give the main result in this section.

THEOREM 7.8. *Let H be a Hermitian $(s_1, s_2, \cdots, s_l)$ block-Hankel matrix with nonzero main skew-diagonal entries $\alpha_1, \ldots, \alpha_l$ and let $t_i = sign(\alpha_i)$, for $i = 1, \ldots, l$. Let*

$$\mathcal{L} = t_1\mathcal{R}_{s_1} \oplus \cdots \oplus t_l\mathcal{R}_{s_l}.$$

Then, there exists a nonsingular upper triangular $(s_1, s_2, \cdots, s_l)$ block-Toeplitz matrix Q such that $Q^*\mathcal{L}Q = H$.

Proof. The proof is by induction on l. Since H is Hermitian and the main diagonal blocks of H are Hankel matrices, these blocks have real entries. Thus, if $l = 1$, the claim follows from Proposition 7.3.

Now suppose that $l > 1$. Let R_{11} be as in (7.3). Partition H as

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},$$

with $H_{22} \in H_{s_l\times s_l}$. Note that H_{22} is real and H_{11} is an $(s_1, s_2, \cdots, s_{l-1})$ block-Hankel matrix with main skew-diagonal entries $\alpha_1, \ldots, \alpha_{l-1}$. By the induction hypothesis, there exists a nonsingular upper triangular $(s_1, s_2, \cdots, s_{l-1})$ block-Toeplitz matrix Q_{11} such that $Q_{11}^*R_{11}Q_{11} = H_{11}$.

Next we find Q_{12} and Q_{22} so that

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{bmatrix}$$

satisfies $Q^*\mathcal{L}Q = H$, or equivalently, taking into account that H is Hermitian,

$$Q_{11}^*R_{11}Q_{12} = H_{12},$$

and

$$Q_{12}^*R_{11}Q_{12} + Q_{22}^*t_l\mathcal{R}_{s_l}Q_{22} = H_{22}. \tag{7.5}$$

Clearly, $Q_{12} := R_{11}Q_{11}^{-*}H_{12}$. By Lemma 7.7, Q_{12} has the form (7.4) with $Q_i \in T^0_{s_i\times s_l}$. Taking into account Proposition 7.6, it follows that $Q_{12}^*R_{11}Q_{12} = \sum_{i=1}^{l-1} Q_i^*t_i\mathcal{R}_{s_i}Q_i \in H^0_{s_l\times s_l}$ and is real. Thus, the existence of $Q_{22} \in T_{s_l\times s_l}$ satisfying (7.5) follows from Proposition 7.3. □

8. Proof of Theorem 4.1. In this section we prove Theorem 4.1. We will need the next lemma, which is a refinement of Proposition 2.4. To prove it we use the following fact, which is a simple consequence of Theorem 4.46 in [12] (see also the exercise after this theorem).

REMARK 8.1. *Let $A = A_1 \oplus A_2$, with $A_1 \in \mathbb{C}^{p\times p}$ and $A_2 \in \mathbb{C}^{q\times q}$. Suppose that A_1 and A_2 have no eigenvalues in common. Then any matrix X commuting with A has the form $X = X_1 \oplus X_2$, with $X_1 \in \mathbb{C}^{p\times p}$ and $X_2 \in \mathbb{C}^{q\times q}$.*

LEMMA 8.2. *Let $P(\lambda)$ be an $n \times n$ Hermitian matrix polynomial as in (1.1) with nonsingular A_k and let (X, J) be a Jordan pair for $P(\lambda)$. Suppose that $J = J_1 \oplus J_2$, where $J_1 = J_{11} \oplus \cdots \oplus J_{1l}$ is of size $p \times p$ and has real eigenvalues, J_2 is of size $q \times q$ and has nonreal eigenvalues, each J_{1i}, $i = 1, \ldots, l$, is a direct sum of Jordan blocks corresponding to the same eigenvalue, and J_{1i} and J_{1j} have distinct eigenvalues for $i \neq j$. Then, the following conditions hold:*

(i) If Z is the (X,J)-matrix, then

$$Z^* B_P Z = H_1 \oplus H_2, \tag{8.1}$$

where $H_1 = H_{11} \oplus \cdots \oplus H_{1l}$ for some H_{1i}, $i = 1, \ldots, l$, is of size equal to the size of J_{1i}, and H_2 is of size $q \times q$.

(ii) For a set of signs ϵ associated with J, $(J, P_{\epsilon,J})$ is a canonical pair for $P(\lambda)$ if and only if there exist nonsingular matrices Q_{1i}, $i = 1, \ldots, l$, such that

$$J_{1i} = Q_{1i}^{-1} J_{1i} Q_{1i}, \quad \text{and} \quad Q_{1i}^* P_{1i} Q_{1i} = H_{1i}, \tag{8.2}$$

where $P_{\epsilon,J} = P_1 \oplus P_2$, $P_1 = P_{11} \oplus \cdots \oplus P_{1l}$ is $p \times p$ and is partitioned as J_1, P_2 is $q \times q$, and $H_{1,i}$ $i = 1, \ldots, l$ are as in (i).

Proof. We first note the following fact that follows from Remark 8.1, taking into account that $J_{11}, \ldots, J_{1l}, J_2$ have distinct eigenvalues, and will be used in the proof: if Q is nonsingular and $Q^{-1}JQ = J$, then

$$Q = Q_{11} \oplus \cdots \oplus Q_{1l} \oplus Q_2, \tag{8.3}$$

where Q_{1i}, $i = 1, \ldots, l$, has the same size as J_{1i}, and Q_2 is $q \times q$.

Then, condition (i) follows from the "only if" claim in Proposition 2.4 (observe that, by Theorem 2.2, there is a canonical pair for $P(\lambda)$). The "only if" claim in (ii) follows with similar arguments.

Now we show the " if" claim in (ii). Suppose that a pair $(J, P_{\epsilon,J})$ is such that (8.2) holds for some nonsingular matrices Q_{1i}, $i = 1, \ldots, l$. We will show that $(J, P_{\epsilon,J})$ is a canonical pair for $P(\lambda)$. Let ϵ' be a set of signs associated with J such that $(J, P_{\epsilon',J})$ is a canonical pair for $P(\lambda)$, and consider the partition $P_{\epsilon',J} = P_1' \oplus P_2'$, where P_1' is $p \times p$ and P_2' is $q \times q$. Note that, because J_2 has no real eigenvalues, J_2 completely determines P_2 and P_2' (recall Definition 2.1). More precisely, we have that $P_2' = P_2$. By Proposition 2.4, if Z is the (X,J)-matrix, we have

$$J = V^{-1} J V \text{ and } V^* P_{\epsilon',J} V = Z^* B_P Z \tag{8.4}$$

for some nonsingular V. Since J_1 and J_2 have no common eigenvalues and J commutes with V, again by Remark 8.1 we have

$$V = V_1 \oplus V_2,$$

where V_1 is $p \times p$ and V_2 is $q \times q$. By condition (i), (8.1) holds. Thus, condition (8.4) is equivalent to

$$V_1^* P_1' V_1 = H_1, \; V_2^* P_2' V_2 = H_2, \; V_1^{-1} J_1 V_1 = J_1, \text{ and } V_2^{-1} J_2 V_2 = J_2.$$

Since (8.2) holds, we have that $J_1 = Q_1^{-1} J_1 Q_1$ and $V_1^* P_1' V_1 = H_1 = Q_1^* P_1 Q_1$, where $Q_1 = Q_{11} \oplus \cdots \oplus Q_{1l}$. Then, taking into account that $P_2' = P_2$,

$$J = W^{-1} J W \quad \text{and} \quad W^* P_{\epsilon,J} W = Z^* B_P Z,$$

where $W = Q_1 \oplus V_2$. By Proposition 2.4, we deduce that $(J, P_{\epsilon,J})$ is a canonical pair for $P(\lambda)$. □

We now prove the main result in this paper.

Proof of Theorem 4.1. Let $(J, P_{\epsilon,J})$ be a canonical pair for $P(\lambda)$ and (X,J) be the associated reducing Jordan pair. Let $L(\lambda) = D(P,v) \in \mathbb{H}(P)$ be a linearization of $P(\lambda)$. Assume that $J = J_1 \oplus J_2$, where J_2 has no real eigenvalues and $J_1 = J_{11} \oplus \cdots \oplus J_{1\alpha}$ has the real eigenvalues of $P(\lambda)$. Moreover, assume that J_{1i} only has one eigenvalue, say λ_i, and $\lambda_i \neq \lambda_j$ if $i \neq j$. Suppose that the main diagonal block of $P_{\epsilon,J}$ corresponding to J_{1i} is

$$P_{1i} := \epsilon_{i,1}\mathcal{R}_{s_{i,1}} \oplus \cdots \oplus \epsilon_{i,l_i}\mathcal{R}_{s_{i,l_i}}.$$

Let $Z(X)$ be the extended matrix associated with X. Then, by Proposition 5.5 and by definition of the matrix $H(X, J, v)$, we have

$$J = Z(X)^{-1}C_L Z(X), \quad Z(X)^* B_L Z(X) = H(X, J, v).$$

By Lemma 8.2 (i) (applied to $L(\lambda)$) and Corollary 6.4, we have $H(X, J, v) = H_{11} \oplus \cdots H_{1\alpha} \oplus H_2$, where, for $i = 1, \ldots, \alpha$, H_{1i} is a Hermitian $(s_{i,1}, \ldots, s_{i,l_i})$ block-Hankel matrix whose main skew-diagonal entries are given by

$$sign(p(\lambda_i; v))\epsilon_{i,1}, \ldots, sign(p(\lambda_i; v))\epsilon_{i,l_i}.$$

Note that, since $L(\lambda)$ is a linearization of $P(\lambda)$, by Theorem 3.1, $p(\lambda_i; v) \neq 0$ for all λ_i.

By Theorem 7.8, for each H_{1i} there exists a nonsingular upper triangular $(s_{i,1}, \ldots, s_{i,l_i})$ block-Toeplitz matrix Q_{1i} such that

$$Q_{1i}^* sign(p(\lambda_i; v))P_{1i}Q_{1i} = H_{1i}.$$

Because of the structure of Q_{1i}, we have $Q_{1i}J_{1i} = J_{1i}Q_{1i}$ (see Lemma 4.4.11 in [12]). Then, by Lemma 8.2 (ii), $(J, P_{\epsilon',J})$ is a canonical pair for $L(\lambda)$, where the main diagonal block of $P_{\epsilon',J}$ corresponding to J_{1i} is

$$P_{1i} := sign(p(\lambda_i; v))[\epsilon_{i,1}\mathcal{R}_{s_{i,1}} \oplus \cdots \oplus \epsilon_{i,l_i}\mathcal{R}_{s_{i,l_i}}].$$

Thus, the claim follows. □

We close this section with the following remark. Let $L(\lambda) = D(P, v) \in \mathbb{H}(P)$ be a linearization of $P(\lambda)$. Using the notation in Lemma 8.2, it follows from this lemma that

$$Z^* B_L Z = H_{11} \oplus \cdots \oplus H_{1\alpha} \oplus H_2,$$

where H_{1i}, $i = 1, \ldots, \alpha$, is of size equal to the size of J_{1i}, that is, corresponds to the blocks in J with the same eigenvalue λ_i, and H_2 corresponds to the blocks in J with nonreal eigenvalues. In Corollary 6.4 we have shown that each matrix H_{1i} is block-Hankel when partitioned as the corresponding block in J. We want to note that we can show that each block H_{1i} is in fact block-diagonal and we can completely characterize the main diagonal blocks by giving the additional description of the entries below the main skew-diagonal. In fact, we can show that, for each of these main diagonal blocks, the entries in the jth diagonal below the main skew-diagonal are given by $\frac{\epsilon_i}{j!}p^{(j)}(\lambda_i; v)$, where ϵ_i is the sign in the sign characteristic of $P(\lambda)$ associated with the corresponding Jordan block in J, λ_i is the associated eigenvalue, and $p^{(j)}(\lambda_i; v)$ denotes the jth derivative of the v-polynomial evaluated at λ_i. Moreover, the matrices Q_{1i} in the proof of Theorem 4.1 can be taken block-diagonal with main diagonal blocks of the form $Q(B)$, for some matrices B. We did not consider this more detailed description of the structure of H as it is not necessary to prove the main result and some of the proofs involved are lengthy and very technical.

9. Conclusions. In this paper, for a Hermitian matrix polynomial $P(\lambda)$ with nonsingular leading coefficient, we have described the sign characteristic of the Hermitian linearizations of $P(\lambda)$ in the family $\mathbb{H}(P)$ of Hermitian pencils in $\mathbb{DL}(P)$, in terms of the sign characteristic of $P(\lambda)$. The connection between the sign characteristic of $P(\lambda)$ and the sign characteristic of its linearizations in $\mathbb{H}(P)$ is provided by the evaluation of the v-polynomial at the real eigenvalues of $P(\lambda)$.

Because the linearizations of $P(\lambda)$ in the family of Hermitian GFPR are $*$-congruent to one of the last two pencils in the standard basis of $\mathbb{DL}(P)$, we also described the sign characteristic of the linearizations of $P(\lambda)$ in that family.

In our study, we have considered the classical definition of the sign characteristic, which only applies to matrix polynomials with nonsingular leading coefficient, that is, to regular matrix polynomials with no infinite eigenvalues. By considering the recent definition of sign characteristic given in [14], in a future work, we intend to extend our results to Hermitian linearizations in $\mathbb{DL}(P)$ when $P(\lambda)$ is a regular Hermitian matrix polynomial with infinite elementary divisors. Note that, for $P(\lambda)$ singular, $\mathbb{DL}(P)$ has no linearizations of $P(\lambda)$. We also plan to study the sign characteristic of Hermitian GFPR linearizations of $P(\lambda)$ when $P(\lambda)$ has singular leading coefficient (that is, $P(\lambda)$ is singular or regular with infinite elementary divisors). In this case an approach different from the one considered in the paper should be followed since $D(P, e_k)$ and $D(P, e_{k-1})$ are not linearizations of $P(\lambda)$.

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