# The Lorentz-cone: spectrum of some classes of matrices and a complete description of the spectral linear presevers in certain subspaces of $M_n$.

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**Abstract.** In this paper we completely characterize the linear maps $\phi : \mathcal{M} \to \mathcal{M}$ that preserve the Lorentz-cone spectrum, when $\mathcal{M}$ is one of the following subspaces of the space $M_n$ of $n \times n$ real matrices: the subspace of diagonal matrices, the subspace of block-diagonal matrices $\widetilde{A} \oplus [a]$, where $\widetilde{A} \in M_{n-1}$ is symmetric, and the subspace of block-diagonal matrices $\widetilde{A} \oplus [a]$, where $\widetilde{A} \in M_{n-1}$ is a generic matrix. In particular, we show that $\phi$ should be what we call a standard map, namely, a map of the form $\phi(A) = PAQ$ for all $A \in \mathcal{M}$ or $\phi(A) = PA^TQ$ for all $A \in \mathcal{M}$, for some matrices $P, Q \in M_n$. We then characterize the standard maps preserving the Lorentz-cone spectrum, when $\mathcal{M}$ is the subspace $S_n$ of symmetric matrices. The case $\mathcal{M} = M_n$ was considered in a recent paper by Seeger (LAA 2020). We include it here for completeness. We conjecture that in both cases $\mathcal{M} = \mathcal{M}_n$ and $\mathcal{M} = \mathcal{S}_n$, the linear preservers of the Lorentz-cone spectrum should be standard maps.

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## 1. Introduction

Given a matrix $A \in M_n$, the algebra of $n \times n$ matrices with real entries, and a closed convex cone $K$, the *eigenvalue complementarity problem* looks for a scalar $\lambda \in \mathbb{R}$ and an $n \times 1$ nonzero vector $x \in \mathbb{R}^n$ such that

$$x \in K, \quad Ax - \lambda x \in K^*, \quad \langle x, Ax - \lambda x \rangle = 0,$$

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where

$$K^* := \{y \in \mathbb{R}^n : \langle x, y\rangle \geq 0, \ \forall x \in K\}$$

denotes the (positive) dual cone of $K$. If $K = \mathbb{R}^n$, then the problem reduces to the usual eigenvalue problem for the matrix $A$. There are, however, a number of interesting applications arising in practice for which $K \neq \mathbb{R}^n$.

Here, we assume $n \geq 3$ and consider the Lorentz cone

$$\mathcal{K}^n := \{(x, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \|x\| \leq x_n\},$$

also known as the ice-cream cone. By $\|x\|$ we denote the 2-norm of $x$. In what follows, the context will make clear what the dimension of the ambient space is and so we omit the superscript $n$ and simply denote $\mathcal{K}^n$ by $\mathcal{K}$. The Lorentz cone is an example of what is called a second-order cone, an object of intense study, especially in optimization theory. The second-order cone is important in linear programming problems, convex quadratic programs and quadratically constrained convex quadratic programs, which in turn, model applications from a variety of fields including engineering, control and finance, as well as in robust optimization and combinatorial optimization. The Lorentz cone is also important from the point of view of understanding certain special linear maps on them, especially the so-called $Z$-transformations. We refer the reader to [1] for applications of the Lorentz cone in optimization, while [5] is a recent work with applications to nonnegative matrix theory.

It is well-known that the Lorentz cone $\mathcal{K}$ is self-dual, that is, $\mathcal{K}^* = \mathcal{K}$. Therefore, for $A \in M_n$, the eigenvalue complementarity problem associated to $\mathcal{K}$ consists of finding a scalar $\lambda \in \mathbb{R}$ and a nonzero vector $x \in \mathbb{R}^n$ such that

$$x \in \mathcal{K}, \quad (A - \lambda I)x \in \mathcal{K}, \quad x^T(A - \lambda I)x = 0, \tag{1.1}$$

where $I$ denotes the identity matrix of the appropriate order. We call the scalar $\lambda$ a *Lorentz eigenvalue* of $A$ and we call $x$ an associated *Lorentz eigenvector* of $A$. We denote by $\sigma_{\mathcal{K}}(A)$ the set of all Lorentz eigenvalues of $A$ and we call it the *Lorentz-cone spectrum of* $A$. By Corollary 2.1 in [6], it is guaranteed that (1.1) admits always a solution.

This article was motivated by the recent work [2], where the authors consider the problem of characterizing the linear preservers of the Pareto-eigenvalues, which is associated with the eigenvalue complementarity problem when the convex cone $K$ is the nonnegative orthant $\mathbb{R}^n_+$. Here we consider the characterization of the linear preservers of the Lorentz-cone spectrum. Let us recall that a linear map $\phi$ is said to be a linear preserver of the Lorentz-cone spectrum if $\sigma_{\mathcal{K}}(\phi(A)) = \sigma_{\mathcal{K}}(A)$, for all $A \in \mathcal{M}$.

Our manuscript consists of two parts. The first part focusses its attention on determining the Lorentz-cone spectrum of some classes of matrices that are relevant in proving the results in the second part, which focusses on characterizing the linear maps $\phi : \mathcal{M} \to \mathcal{M}$ that preserve the Lorentz-cone spectrum, for some subspaces $\mathcal{M}$ of $M_n$. We first consider the following subspaces $\mathcal{M}$ : the subspace of diagonal matrices; the subspace of block-diagonal matrices $\widetilde{A} \oplus [a]$, where $\widetilde{A} \in M_{n-1}$ is symmetric; and the subspace of block-diagonal matrices $\widetilde{A} \oplus [a]$, where $\widetilde{A} \in M_{n-1}$ is a generic matrix. In each of these cases, we completely characterize the linear preservers of the Lorentz-cone spectrum. In particular, we show that they should be what we call standard maps, that is, maps of the form $\phi(A) = PAQ$ for all $A \in \mathcal{M}$ or $\phi(A) = PA^TQ$ for all $A \in \mathcal{M}$, for some matrices $P, Q \in M_n$. (By $A^T$ we denote the transpose of $A$.) We then characterize the Lorentz-cone spectrum preservers that are standard maps when $\mathcal{M}$ is the subspace $S_n$ of symmetric matrices in $M_n$ and $\mathcal{M}$ is $M_n$. Very recently, we found the paper [8], in which the result for $M_n$ is proven using different techniques. For completeness we include it here with our proof, hoping that it provides tools that may be helpful in proving our conjecture that all Lorentz-cone spectrum preservers in $M_n$ are standard maps.

## 2. Properties of Lorentz eigenvalues

We next give some simple facts related to the Lorentz-cone and the Lorentz-cone spectrum of a matrix $A \in M_n$. The proofs of these results are easy and, therefore, are omitted. The first result can be helpful in the proof of the second one. See also [8], in which some of these properties appear.

**Proposition 2.1.** *Let $x \in \mathcal{K}$. Then:*

1. *$Px \in \mathcal{K}$ for any matrix $P$ of the form*
$$P = \begin{bmatrix} \tilde{P} & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.1}$$
*where $\tilde{P} \in M_{n-1}$ is orthogonal.*
2. *$Dx \in \mathcal{K}$ for any diagonal matrix $D = diag(d_1, d_2, \ldots, d_n) \in M_n$, with $d_n \geq 0$ and $|d_i| \leq d_n$, $i = 1, 2, \ldots, n-1$. In particular, $\gamma x \in \mathcal{K}$ for any $\gamma \geq 0$.*

**Proposition 2.2.** *Let $A \in M_n$. Then,*

1. *$\sigma_{\mathcal{K}}(\gamma A) = \gamma\sigma_{\mathcal{K}}(A)$, for all $\gamma \geq 0$.*
2. *$\sigma_{\mathcal{K}}(A + \gamma I) = \sigma_{\mathcal{K}}(A) + \gamma$, for all $\gamma \in \mathbb{R}$.*
3. *$\sigma_{\mathcal{K}}(PAP^{-1}) = \sigma_{\mathcal{K}}(A)$, for any invertible matrix $P \in M_n$ of the form (2.1), where $\tilde{P} \in M_{n-1}$ is an orthogonal matrix.*
4. *$\sigma_{\mathcal{K}}(A) \subseteq \sigma_{\mathcal{K}}(DAD^{-1})$, for any diagonal matrix $D = diag(d_1, \ldots, d_n)$ with $d_n > 0$ and $|d_i| \leq d_n$, $i = 1, 2, \ldots, n-1$. Moreover, equality holds in the inclusion above, if $D$ is a positive multiple of a signature matrix with $d_n = 1$.*
5. *$\sigma_{\mathcal{K}^m}(A_2) \subseteq \sigma_{\mathcal{K}^n}(A)$, for any matrix $A$ of the form $A = A_1 \oplus A_2$ with $A_1 \in M_{n-m}$ and $A_2 \in M_m$.*

We next give a characterization of Lorentz eigenvalues, alternative to the definition, which has shown to be very convenient in practice. Notice that, using the standard topology in $\mathbb{R}^n$, a vector $x = [z^T\ x_n]^T \in \mathcal{K}$, where $z \in \mathbb{R}^{n-1}$, is in the interior of $\mathcal{K}$ if and only if $\|z\| < x_n$. Also, for any $\lambda \in \sigma_{\mathcal{K}}(A)$, there is an associated Lorentz eigenvector with the last entry equal to 1.

The Lorentz-cone spectrum of $A \in M_n$ can be seen as the union of two (not necessarily disjoint) sets:

$$\sigma_{\mathcal{K}}(A) = \sigma_{int}(A, \mathcal{K}) \cup \sigma_{bd}(A, \mathcal{K}),$$

where $\sigma_{int}(A, \mathcal{K})$ contains the Lorentz eigenvalues for which there exists an associated Lorentz eigenvector in the interior of $\mathcal{K}$ and $\sigma_{bd}(A, \mathcal{K})$ contains the Lorentz eigenvalues for which there exists an associated Lorentz eigenvector on the boundary of $\mathcal{K}$.

It is known [7] that $\lambda \in \sigma_{int}(A, \mathcal{K})$ if and only if $\lambda$ is a standard eigenvalue of $A$ (that is, $\lambda \in \sigma(A)$), which, however, is associated with an eigenvector in the interior of $\mathcal{K}$.

By [7, Proof of Theorem 4.2], we have that $\lambda \in \sigma_{bd}(A, \mathcal{K})$ if and only if there exists $z \in \mathbb{R}^{n-1}$, with $\|z\| = 1$, and $s, \mu \in \mathbb{R}$, with $s > 0$, such that $\lambda = \mu + s$ and

$$\begin{bmatrix} \widetilde{A} - \mu I & u \\ v^T & a - \mu - 2s \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = 0, \tag{2.2}$$

with $\widetilde{A} \in M_{n-1}$.

For our purposes, it will be convenient to distinguish between the Lorentz eigenvalues of a matrix that are also eigenvalues in the usual sense, and those that are not. We say that $\lambda \in \sigma_{\mathcal{K}}(A)$ is a *standard Lorentz eigenvalue* of $A \in M_n$ if $\lambda \in \sigma(A)$. We say that $\lambda$ is a *nonstandard Lorentz eigenvalue* of $A$ otherwise.

Note that in the characterization of boundary Lorentz eigenvalues given above, if $s = 0$, then $\lambda = \mu$ is a standard Lorentz eigenvalue.

An interesting feature of the Lorentz-cone spectrum of a matrix $A \in M_n$, or, more precisely, of $\sigma_{bd}(A, \mathcal{K})$, is that it may be infinite, in stark contrast with the standard spectrum. Observe that $\sigma_{int}(A, \mathcal{K})$ is always finite, since it is a subset of the standard spectrum of $A$. The next lemma gives necessary conditions for a matrix to have infinitely many Lorentz eigenvalues. Its proof follows from [7, Theorem 4.2 and the proof of Corollary 4.4].

**Lemma 2.3.** *Let*

$$A = \begin{bmatrix} \widetilde{A} & u \\ v^T & a \end{bmatrix} \in M_n.$$

*If $A$ has infinitely many Lorentz eigenvalues, then there exist $\mu \in \sigma(\widetilde{A})$ with geometric multiplicity at least 2 and $s > 0$ such that:*

1. $u \in Im(\widetilde{A} - \mu I)$,
2. $v \notin Im(\widetilde{A}^T - \mu I)$,
3. $\left\| \begin{bmatrix} \widetilde{A} - \mu I \\ v^T \end{bmatrix}^{\dagger} \begin{bmatrix} u \\ a - \mu - 2s \end{bmatrix} \right\| \leq 1,$

   *where $B^\dagger$ denotes the Moore-Penrose inverse of the matrix $B$.*

## 3. Lorentz eigenvalues of special matrices.

In this section, we determine the Lorentz-cone spectrum of some structured matrices. These results will be used in the sequel.

**Theorem 3.1.** *Let*

$$A = \widetilde{A} \oplus [a] \in M_n,$$

*with* $a \in \mathbb{R}$. *Then,*

$$\sigma_{\mathcal{K}}(A) = \{a\} \cup \left\{ \frac{\mu + a}{2} : \mu \in \sigma(\widetilde{A}),\ \mu < a \right\}.$$

*Moreover,* $a$ *is the only standard Lorentz eigenvalue of* $A$.

*Proof.* It can be easily verified that there exists $\lambda \in \mathbb{R}$ and a vector $x \in \mathbb{R}^{n-1}$ such that

$$\left( \widetilde{A} \oplus [a] - \lambda I \right) \begin{bmatrix} x \\ 1 \end{bmatrix} = 0, \tag{3.1}$$

if and only if, $\lambda = a$. Moreover, (3.1) holds for $x = 0$ and $\lambda = a$. Thus, $a$ is the only standard Lorentz eigenvalue of $A$.

Next, suppose that $\lambda$ is a nonstandard Lorentz eigenvalue of $A$. Then, there are $\mu, s \in \mathbb{R}$, with $s > 0$ and $\lambda = \mu + s$, and a vector $z \in \mathbb{R}^{n-1}$, with $\|z\| = 1$, such that

$$\begin{bmatrix} \widetilde{A} - \mu I & 0 \\ 0 & a - \mu - 2s \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = 0. \tag{3.2}$$

This implies that $\frac{a-\mu}{2} = s > 0$ and $(\widetilde{A} - \mu I)z = 0$. Thus, $\mu < a$, $\mu$ is an eigenvalue of $\widetilde{A}$ and $\lambda = s + \mu = \frac{\mu+a}{2}$. Conversely, suppose that $\lambda = \frac{\mu+a}{2}$, in which $\mu$ is an eigenvalue of $\widetilde{A}$ and $\mu < a$. Let $z \in \mathbb{R}^{n-1}$ be a unit eigenvector of $\widetilde{A}$ associated with $\mu$. Thus, for $s = \lambda - \mu = \frac{a-\mu}{2} > 0$, (3.2) holds, implying that $\lambda$ is a nonstandard Lorentz eigenvalue of $A$. □

Note that, if $A$ is the zero matrix, then 0 is the only Lorentz eigenvalue of $A$.

**Corollary 3.2.** *Let*

$$A = \widetilde{A} \oplus [a] \in M_n$$

*and*

$$B = \widetilde{B} \oplus [a] \in M_n,$$

*with* $a \in \mathbb{R}$. *If* $\widetilde{A}$ *and* $\widetilde{B}$ *are similar matrices, then* $\sigma_{\mathcal{K}}(A) = \sigma_{\mathcal{K}}(B)$.

**Corollary 3.3.** *Let* $D = diag(d_1, d_2, \ldots, d_n) \in M_n$ *be a diagonal matrix. Then,*

$$\sigma_{\mathcal{K}}(D) = \{d_n\} \cup \left\{ \frac{d_i + d_n}{2} : d_i < d_n \right\}.$$

In the next result, we study the Lorentz-cone spectrum of another class of matrices, obtained by bordering the zero matrix, which does not include diagonal matrices.

**Theorem 3.4.** *Let*

$$A = \begin{bmatrix} 0 & u \\ v^T & a \end{bmatrix},$$

*with $u, v \in \mathbb{R}^{n-1}$ not both zero, and $a \in \mathbb{R}$. Then,*

1. *$0$ is a standard Lorentz eigenvalue of $A$ if, and only if, $u = 0$ and $|a| \leq \|v\|$.*
2. *If $\lambda \neq 0$, then $\lambda$ is a standard Lorentz eigenvalue of $A$ if, and only if, $|\lambda| \geq \|u\|$ and $\lambda^2 - a\lambda - v^T u = 0$.*
3. *If $u \neq 0$, then $\lambda$ is a nonstandard Lorentz eigenvalue of $A$ if, and only if, one of the following holds:*
   (i) *$v^T u + a\|u\| - \|u\|^2 > 0$ and $\lambda = \frac{1}{2\|u\|}[a\|u\| + \|u\|^2 + v^T u]$.*
   (ii) *$\|u\|^2 + a\|u\| - v^T u > 0$ and $\lambda = \frac{1}{2\|u\|}[a\|u\| - \|u\|^2 - v^T u]$.*
4. *If $u = 0$ ($v \neq 0$), then $\lambda$ is a nonstandard Lorentz eigenvalue of $A$ if and only if*

$$\lambda \in \left[\frac{a - \|v\|}{2}, \frac{a + \|v\|}{2}\right] \cap (0, \infty).$$

*Proof.* To prove 1. and 2., we observe that a constant $\lambda$ is a standard Lorentz eigenvalue of $A$ if, and only if, there exists a vector $x \in \mathbb{R}^{n-1}$ such that $\|x\| \leq 1$ and

$$\begin{bmatrix} -\lambda & u \\ v^T & a - \lambda \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = 0. \tag{3.3}$$

**Claim 1**: From the equation as above, $\lambda = 0$ is a standard eigenvalue of $A$ if, and only if, $u = 0$ and $v^T x + a = 0$ has a solution in the unit disk. This, in turn, is equivalent to $|a| \leq \|v\|$.

**Claim 2**: A scalar $\lambda \neq 0$ is a standard eigenvalue of $A$ if, and only if, there is a solution $x$ in the unit disk for the system

$$x = \frac{u}{\lambda}, \quad v^T x + a - \lambda = 0.$$

The requirement that $\|x\| \leq 1$, is equivalent to $\|u\| \leq |\lambda|$. Replacing the value of $x$ in the second condition, one obtains the required inequality.

To prove 3. and 4., we observe that a constant $\lambda$ is a nonstandard Lorentz eigenvalue of $A$ if, and only if, $\lambda = \mu + s$, with $s > 0$, and

$$\begin{bmatrix} -\mu & u \\ v^T & a - \mu - 2s \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = 0, \tag{3.4}$$

for some vector $x \in \mathbb{R}^{n-1}$ with $\|x\| = 1$. The identity (3.4) is equivalent to

$$-\mu x + u = 0 \quad \text{and} \quad v^T x + a - \mu - 2s = 0. \tag{3.5}$$

**Claim 3**: The conditions in (3.5) are equivalent to $x = \frac{u}{\mu}$ and $s = \frac{v^T x + a - \mu}{2}$. Note that, since $u \neq 0$, also $\mu \neq 0$. We have $\|x\| = 1$ if, and only if, $|\mu| = \|u\|$ and so one has $\mu = \|u\|$ or $\mu = -\|u\|$. By replacing the corresponding vectors $x$ in the expression for $s$, conclusions (i) and (ii) follow.

**Claim 4**: When $u = 0$, so that $\mu = 0$, one has $\lambda = s > 0$. Moreover, there exists a vector $x$, with $\|x\| = 1$, for which $v^T x + a - 2s = 0$ if, and only if, $\frac{|a-2s|}{\|v\|} = \frac{|a-2\lambda|}{\|v\|} \leq 1$. Thus,

$$\lambda \in \left[\frac{a-\|v\|}{2}, \frac{a+\|v\|}{2}\right] \cap (0, \infty).$$

□

As a consequence of Corollary 3.3 and Theorem 3.1, we have the following result:

**Corollary 3.5.** *Let $E_{ij} := e_i^T e_j \in M_n$, where $e_i$ denotes the ith standard basis vector in $\mathbb{R}^n$. We then have the following:*
*(i) For each $i = 1, 2, \cdots, n-1$,*

$$\sigma_{\mathcal{K}}(E_{ni}) = [0, 1/2], \qquad \sigma_{\mathcal{K}}(E_{in}) = \{-1/2\}.$$

*and*

$$\sigma_{\mathcal{K}}(E_{ni} + E_{in}) = \{-1, 1\}.$$

*(ii) For $i, j \in \{1, 2, \ldots, n-1\}$, with $i \neq j$,*

$$\sigma_{\mathcal{K}}(E_{nj} + E_{in}) = \{-1/2\}.$$

*(iii) For each $i = 1, 2, \ldots, n-1$,*

$$\sigma_{\mathcal{K}}(E_{ii}) = \{0\}, \quad \text{and} \quad \sigma_{\mathcal{K}}(E_{nn}) = \{1, 1/2\}.$$

Next, we describe the Lorentz-cone spectrum of symmetric rank-one matrices. First, we recall a well-known result for the inverse of a rank-one perturbed invertible matrix, that will be used in its proof.

**Lemma 3.6.** *(Sherman-Morrison Formula) Suppose $A \in M_n$ is an invertible matrix and $u, v \in \mathbb{R}^n$ are column vectors. Then, $A + uv^T$ is invertible if and only if $1 + v^T A^{-1} u \neq 0$. In this case*

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1} u v^T A^{-1}}{1 + v^T A^{-1} u}.$$

**Theorem 3.7.** *Let $\alpha, a \in \mathbb{R}$ with $\alpha \neq 0$, and let $0 \neq y = [\hat{y}^T \ a]^T \in \mathbb{R}^n$ be a unit vector. Let*

$$A = \alpha y y^T = \left[\begin{array}{c|c} \alpha \hat{y}\hat{y}^T & \alpha a \hat{y} \\ \hline \alpha a \hat{y}^T & \alpha a^2 \end{array}\right].$$

*Then, the Lorentz-cone spectrum of $A$ is given as follows:*

1. *if $a = 0$ and $\hat{y} \neq 0$, then* $\sigma_{\mathcal{K}}(A) = \begin{cases} \{0\}, & \text{if } \alpha > 0 \\ \{0, \alpha\|\hat{y}\|^2/2\}, & \text{if } \alpha < 0 \end{cases};$
2. *if $a \neq 0$ and $\hat{y} = 0$, then* $\sigma_{\mathcal{K}}(A) = \begin{cases} \{\alpha a^2\}, & \text{if } \alpha < 0 \\ \{\alpha a^2, \alpha a^2/2\}, & \text{if } \alpha > 0 \end{cases};$

3. *if* $a \neq 0$ *and* $\hat{y} \neq 0$, *then*

$$\sigma_{\mathcal{K}}(A) = \begin{cases} \{0, \alpha\}, & \text{if } |a| = \|\hat{y}\| \\ \{0\}, & \text{if } |a| < \|\hat{y}\| \text{ and } \alpha > 0 \\ \{0, \alpha\frac{(a \pm \|\hat{y}\|)^2}{2}\}, & \text{if } |a| < \|\hat{y}\| \text{ and } \alpha < 0 \\ \{\alpha\}, & \text{if } |a| > \|\hat{y}\| \text{ and } \alpha < 0 \\ \{\alpha, \alpha\frac{(a \pm \|\hat{y}\|)^2}{2}\}, & \text{if } |a| > \|\hat{y}\| \text{ and } \alpha > 0 \end{cases}.$$

*Proof.* When $a = 0$ and $\hat{y} \neq 0$, or when $a \neq 0$ and $\hat{y} = 0$, the conclusions follow from Theorem 3.1.
Next, assume that $a \neq 0$ and $\hat{y} \neq 0$.

**Case 1** (Standard Lorentz eigenvalues): Since the (standard) eigenvalues of $A$ are 0 and $\alpha$, these are the only candidates for standard Lorentz eigenvalues of $A$.

Zero is a standard Lorentz eigenvalue of $A$ if, and only if, there exists a vector $x \in \mathbb{R}^{n-1}$ with $\|x\| \leq 1$ such that

$$0 = \hat{y}\hat{y}^T x + a\hat{y} = \hat{y}(\hat{y}^T x + a) \quad \text{and} \quad 0 = a\hat{y}^T x + a^2 = a(\hat{y}^T x + a). \tag{3.6}$$

Since $\hat{y} \neq 0$ and $a \neq 0$, (3.6) is equivalent to $\hat{y}^T x + a = 0$, which has a solution in the unit disk if, and only if, $|a| \leq \|\hat{y}\|$.

Similarly, $\alpha$ is a standard Lorentz eigenvalue of $A$ if and only if there exists a vector $x \in \mathbb{R}^{n-1}$ with $\|x\| \leq 1$ such that

$$(\hat{y}\hat{y}^T - I)x + a\hat{y} = 0 \quad \text{and} \quad a\hat{y}^T x + a^2 - 1 = 0. \tag{3.7}$$

Since $\|y\| = 1$, we have $\|\hat{y}\|^2 + a^2 = 1$, and (3.7) is equivalent to

$$(\hat{y}\hat{y}^T - I)x + a\hat{y} = 0 \quad \text{and} \quad a\hat{y}^T x - \hat{y}^T\hat{y} = 0. \tag{3.8}$$

Note that the second equation in (3.8) has a solution $x$ in the unit disk if, and only if, $\|\hat{y}\| \leq |a|$, and that $x = \frac{\hat{y}}{a}$ is one of those solutions. Taking into account that $\|y\| = 1$, it can be easily seen that such $x$ is also a solution of the first equation in (3.8).

**Case 2** (Nonstandard Lorentz eigenvalues): We have that $\lambda$ is a nonstandard Lorentz eigenvalue of $A$ if, and only if, there exist $w \in \mathbb{R}^{n-1}$, with $\|w\| = 1$, and $\mu, s \in \mathbb{R}$, with $s > 0$, such that $\lambda = \mu + s$ and

$$(\alpha\hat{y}\hat{y}^T - \mu I)w + \alpha a\hat{y} = 0 \quad \text{and} \quad \alpha a\hat{y}^T w + \alpha a^2 - \mu - 2s = 0. \tag{3.9}$$

Note that the matrix $\alpha\hat{y}\hat{y}^T - \mu I$ is invertible if, and only if, $\mu \notin \{0, \alpha\|\hat{y}\|^2\}$.
Now, we claim that $\mu \notin \{0, \alpha\|\hat{y}\|^2\}$.
Proof of the claim: Suppose that $\mu = 0$. The second condition in (3.9) is equivalent to

$$\hat{y}^T w = \frac{2s - \alpha a^2}{\alpha a}.$$

Replacing this expression in the first equation in (3.9), we have

$$0 = \hat{y}\frac{2s - \alpha a^2}{\alpha a} + a\hat{y} = \frac{2s}{\alpha a}\hat{y},$$

a contradiction, since $s \neq 0$, proving that, $\mu \neq 0$.
If, on the other hand, one has $\mu = \alpha\|\hat{y}\|^2$, then, multiplying on the left

both sides of the first equation in (3.9) by $\hat{y}^T$, we get $\alpha a\|\hat{y}\|^2 = 0$, which is impossible, by the hypothesis. This completes the proof of the claim.

Hence, the matrix $\alpha\hat{y}\hat{y}^T - \mu I$ is invertible and, by Lemma 3.6, the first condition in (3.9) is equivalent to

$$w = (\mu I - \alpha\hat{y}\hat{y}^T)^{-1}\alpha a\hat{y} = \frac{1}{\mu}(I + \frac{\alpha\hat{y}\hat{y}^T}{\mu - \alpha\|\hat{y}\|^2})\alpha a\hat{y} = \frac{\alpha a\hat{y}}{\mu - \alpha\|\hat{y}\|^2}.$$

Thus, $\|w\| = 1$ if and only if $\mu - \alpha\|\hat{y}\|^2 = \alpha a\|\hat{y}\|$ or $\mu - \alpha\|\hat{y}\|^2 = -\alpha a\|\hat{y}\|$. Replacing the obtained formula for $w$ in the second condition in (3.9), we have

$$\frac{\alpha^2 a^2\|\hat{y}\|^2}{\mu - \alpha\|\hat{y}\|^2} + \alpha a^2 - \mu - 2s = 0.$$

For both values of $\mu$, we get $s = \alpha\frac{a^2 - \|\hat{y}\|^2}{2}$. We have $s > 0$ if, and only if, $\alpha > 0$ and $a^2 > \|\hat{y}\|^2$; or $\alpha < 0$ and $a^2 < \|\hat{y}\|^2$. The corresponding eigenvalues $\lambda = \mu + s$ are $\alpha\frac{(a+\|\hat{y}\|)^2}{2}$ and $\alpha\frac{(a-\|\hat{y}\|)^2}{2}$. □

We finally state a result that will be used in Section 4.4 concerning rank-one nilpotent matrices.

**Lemma 3.8.** *Let $A \in M_n$ be a nilpotent rank-one matrix. If $A$ has infinitely many Lorentz eigenvalues, then there exists a nonzero vector $v \in \mathbb{R}^{n-1}$ such that*

$$A = \begin{bmatrix} 0 & 0 \\ v^T & 0 \end{bmatrix}.$$

*Proof.* Since $A$ is a rank-one matrix, there exist vectors $p^T = [\hat{p}^T \ a]$ and $q^T = [\hat{q}^T \ b] \in \mathbb{R}^n$, with $a, b \in \mathbb{R}$, such that

$$A = pq^T = \begin{bmatrix} \hat{p}\hat{q}^T & b\hat{p} \\ a\hat{q}^T & ab \end{bmatrix}.$$

Since $A$ has infinitely many eigenvalues, by Lemma 2.3, there exist $\mu, s \in \mathbb{R}$, with $s > 0$, such that (i) $\mu$ is an eigenvalue of $\hat{p}\hat{q}^T$ with geometric multiplicity at least 2 and (ii) $a\hat{q} \notin \mathrm{Im}(\hat{q}\hat{p}^T - \mu I)$. Since $\hat{p}\hat{q}^T$ is a rank one matrix, the first requirement implies that $\mu = 0$. Since $\mathrm{Im}(\hat{q}\hat{p}^T) = \mathrm{span}\{\hat{\mathrm{q}}\}$, it follows from the second requirement that $\hat{p} = 0$, $a \neq 0$ and $\hat{q} \neq 0$. Moreover, since $A$ is nilpotent, one has $tr(A) = 0$ and so $b = 0$. □

## 4. Lorentz-cone spectrum linear preservers

In this section, we use the following notation:

- $D_n$ denotes the space of $n \times n$ diagonal matrices in $M_n$,
- $L_n := \{\widetilde{A} \oplus [a] \in M_n\colon \widetilde{A} \in M_{n-1} \text{ is symmetric}\}$,
- $W_n := \{\widetilde{A} \oplus [a] \in M_n : \widetilde{A} \in M_{n-1}\}$,
- $S_n$ denotes the space of $n \times n$ symmetric matrices in $M_n$.

In this section, we characterize linear maps $\phi : \mathcal{M} \to \mathcal{M}$ that preserve the Lorentz-cone spectrum, when $\mathcal{M} \in \{D_n, L_n, W_n, S_n, M_n\}$. For $\mathcal{M} \in \{D_n, L_n, W_n\}$ we give a complete characterization of such maps $\phi$, while for $\mathcal{M} \in \{S_n, M_n\}$ we will restrict our attention to Lorentz-cone spectrum preservers that are standard maps, according to the next definition.

**Definition 4.1.** Let $\phi : S \to S$ be a linear map, where $S$ is a subspace of $M_n$. If there exist matrices $P, Q \in M_n$ such that

$$\phi(A) = PAQ \quad \text{for all } A \in M_n \tag{4.1}$$

or

$$\phi(A) = PA^TQ \quad \text{for all } A \in M_n, \tag{4.2}$$

then, we say that $\phi$ is a *standard linear map*. If, in addition, $\phi$ preserves the Lorentz-cone spectrum, we say that $\phi$ is a standard Lorentz-cone spectrum preserver.

Note that, if, in Definition 4.1, $S \subseteq S_n$, then (4.1) and (4.2) coincide.

### 4.1. General properties

In this section we state and prove some important lemmas that will allow us to obtain our main results.

**Lemma 4.2.** *Let $\mathcal{M}$ be any subspace of $M_n$ and $\phi : \mathcal{M} \to \mathcal{M}$ be a linear map preserving the Lorentz-cone spectrum. Assume that $\phi$ is bijective. Then, $\phi^{-1}$ also preserves the Lorentz-cone spectrum.*

*Proof.* Let $A \in \mathcal{M}$. Then,

$$\sigma_{\mathcal{K}}(\phi^{-1}(A)) = \sigma_{\mathcal{K}}(\phi(\phi^{-1}(A))) = \sigma_{\mathcal{K}}(A).$$

$\square$

Our next goal is to show that linear maps $\phi : \mathcal{M} \to \mathcal{M}$, with $\mathcal{M} \in \{D_n, L_n, W_n, S_n, M_n\}$, preserving the Lorentz-cone spectrum are bijective. For this purpose, we need the following result, which is a particular case of the so called "cancellation law" (see [8, Definition 1]). We notice that, when the considered space $\mathcal{M}$ is $M_n$, this result has already been obtained in [8], although using different techniques.

**Lemma 4.3.** *Let $\mathcal{M} \in \{D_n, L_n, W_n, S_n, M_n\}$ and $A \in \mathcal{M}$. If*

$$\sigma_{\mathcal{K}}(A + B) = \sigma_{\mathcal{K}}(B), \quad \textit{for all } B \in \mathcal{M} \tag{4.3}$$

*then $A = 0$.*

*Proof.* Let $A \in \mathcal{M}$ satisfy (4.3). Then,

$$\sigma_{\mathcal{K}}(A) = \sigma_{\mathcal{K}}(A + 0) = \sigma_{\mathcal{K}}(0) = \{0\}, \tag{4.4}$$

where the second equality follows from our assumption and the last equality follows from Theorem 3.1.

**Case 1**: Suppose that $\mathcal{M} \in \{D_n, L_n, W_n\}$. By Theorem 3.1, (4.4) implies that $A = \widetilde{A} \oplus [0]$ and the real eigenvalues of $\widetilde{A}$ are nonnegative. If $\widetilde{A}$ has a positive real eigenvalue $\alpha$, then, for the choice $B = -\widetilde{A} \oplus [0]$, we have that $\frac{-\alpha}{2} \in \sigma_{\mathcal{K}}(B)$ and $\frac{-\alpha}{2} \notin \sigma_{\mathcal{K}}(A + B) = \{0\}$, a contradiction. Thus, the real eigenvalues of $\widetilde{A} \in M_{n-1}$ are zero. If $\mathcal{M} \in \{D_n, L_n\}$ then $A = 0$, as desired. Now, consider $\mathcal{M} = W_n$. Then, there is an orthogonal matrix $Q \in M_{n-1}$ such that

$$\widetilde{C} := Q^T \widetilde{A} Q = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix},$$

where $C_{11} \in M_{p_1}$, if nonempty, is upper triangular with 0 main diagonal, and $C_{22} \in M_{p_2}$, if nonempty, has no real eigenvalues.

Suppose that $p_2 > 0$. Let $\widetilde{B} = Q^T(0_{p_1 \times p_1} \oplus \delta I_{p_2})Q$, with $\delta < 0$. Then, by Theorem 3.1, $\sigma_{\mathcal{K}}(A + (\widetilde{B} \oplus [0])) = \{0\}$ and $\sigma_{\mathcal{K}}(\widetilde{B} \oplus [0]) = \{-\frac{\delta}{2}, 0\}$, a contradiction.

Now suppose that $p_2 = 0$, that is, $\widetilde{A}$ is nilpotent, and $\widetilde{A} \neq 0$. In this case there is a $2 \times 2$ principal submatrix of $\widetilde{C}$ of the form

$$\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix},$$

with $x \neq 0$. Let $\widetilde{F}$ be the matrix obtained from $\widetilde{C}$ by replacing this $2 \times 2$ submatrix by

$$\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}$$

and all the other entries by 0. Thus, $\sigma_{\mathcal{K}}(A + (Q\widetilde{F}Q^T \oplus [0])) = \{0, -\frac{|x|}{2}\}$ and $\sigma_{\mathcal{K}}(Q\widetilde{F}Q^T \oplus [0]) = \{0\}$, a contradiction. Thus, $A = 0$.

**Case 2**: Suppose that $\mathcal{M} \in S_n$. Let

$$A = \begin{bmatrix} \widetilde{A} & u \\ u^T & a \end{bmatrix} \in S_n,$$

with $u \in \mathbb{R}^{n-1}$. Suppose that $u \neq 0$ and $a \neq -2\|u\|$. Let

$$B_1 = \begin{bmatrix} 0 & -u \\ -u^T & -a \end{bmatrix}$$

By Theorem 3.1, $0 \in \sigma_{\mathcal{K}}(A + B_1)$ and, by Theorem 3.4, $0 \notin \sigma_{\mathcal{K}}(B_1)$, a contradiction. Now suppose that $u \neq 0$ and $a = -2\|u\|$. Let

$$B_2 = \begin{bmatrix} 0 & -u \\ -u^T & 0 \end{bmatrix}$$

By Theorem 3.1, $a \in \sigma_{\mathcal{K}}(A + B_2)$ and, by Theorem 3.4, $a \notin \sigma_{\mathcal{K}}(B_2)$, a contradiction.

Thus, $u = 0$ implying that $A \in W_n$ and the proof follows as in Case 1.

**Case 3**: Suppose that $\mathcal{M} \in M_n$. This case can be proved using Theorems 3.1 and 3.4, with arguments similar to those in Case 2. Also, it is shown in [8] using a different approach. Therefore, we omit the proof here. □

The next theorem is a consequence of Lemma 4.3 and provides special properties of the linear maps preserving the Lorentz-cone spectrum. We omit its proof since the deduction from Lemma 4.3 is similar to the one in [8, Proposition 9] (see also [2]). Let us recall that a map $\eta : \mathcal{M} \to \mathcal{M}$ is called *unital* if $\eta(I) = I$.

**Theorem 4.4.** *Let $\mathcal{M} \in \{D_n, L_n, W_n, S_n, M_n\}$. If $\phi : \mathcal{M} \to \mathcal{M}$ is a linear map that preserves the Lorentz-cone spectrum, then $\phi$ is bijective and unital.*

### 4.2. The block-diagonal case

In this section we focus on linear preservers $\phi : \mathcal{M} \to \mathcal{M}$ of the Lorentz-cone spectrum, with $\mathcal{M} \in \{D_n, L_n, W_n\}$. The next lemma will be crucial in characterizing such maps $\phi$, which will be done in Sections 4.2.1 and 4.2.2. It compares the real spectra of $A \in \mathcal{M}$ and $\phi(A)$, as well as those of their $(n-1) \times (n-1)$ leading principal submatrices. Since the claim is valid when $\mathcal{M}$ is any subspace of $W_n$, we state it in this more general form.

**Lemma 4.5.** *Let $\mathcal{M}$ be a subspace of $W_n$ and $\phi : \mathcal{M} \to \mathcal{M}$ be a linear map that preserves the Lorentz-cone spectrum. Let $A = \widetilde{A} \oplus [a] \in \mathcal{M}$ and $\phi(A) = \widetilde{B} \oplus [b]$ . Then, $a = b$ and $\sigma_{\mathbb{R}}(\widetilde{A}) = \sigma_{\mathbb{R}}(\widetilde{B})$, counting multiplicities.*

*Proof.* Let $A$ and $\phi(A)$ be as in the statement. By item (2) of Proposition 2.2, we may assume $a = 0$. First, we show that $b = 0$. Assume $b \neq 0$. Since 0 is a Lorentz eigenvalue of $A$, and therefore of $\phi(A)$, by Theorem 3.1, we have that $0 = \frac{\gamma+b}{2}$ for some $\gamma \in \sigma(\widetilde{B})$ such that $\gamma < b$. Analogously, since 0 is also a Lorentz eigenvalue of $-A$, we have that $0 = \frac{\delta-b}{2}$, where $\delta < -b$ and $-\delta \in \sigma(\widetilde{B})$. Thus, we have

$$\gamma = -b = -\delta \quad \text{and} \quad \gamma < b < -\delta,$$

a contradiction. Thus, $b = 0$.

Suppose that $\widetilde{A}$ has $s \leq n-1$ real eigenvalues (counting multiplicities). Let $\widetilde{T} = Z^T \widetilde{A} Z$ be the real Schur's form of $\widetilde{A}$, where $Z \in M_{n-1}$ is orthogonal and $\widetilde{T}$ is quasi-upper triangular, that is, $\widetilde{T}$ is a block upper-triangular matrix with $1 \times 1$ and $2 \times 2$ blocks on the main diagonal. Without loss of generality, suppose that the $s$ $1 \times 1$ blocks (which correspond to the real eigenvalues) occur in the first $s$ main diagonal positions. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_s)$ be a sequence of distinct positive real numbers. Let us denote by $\widetilde{T}_\varepsilon$ the matrix obtained from $\widetilde{T}$ by adding to $\widetilde{T}_{ii}$ (the main diagonal entry of $\widetilde{T}$ in position $(i,i)$) the real number $\varepsilon_i$, $i = 1, 2, \ldots, s$. Then, if $\|\varepsilon\|$ is small enough, the real eigenvalues of $\widetilde{T}_\epsilon$ are nonzero and distinct. Moreover, $\lim_{\varepsilon \to 0} \widetilde{T}_\varepsilon = \widetilde{T}$. Let $\widetilde{A}_\varepsilon = Z \widetilde{T}_\varepsilon Z^T$ and $A_\varepsilon = \widetilde{A_\varepsilon} \oplus [0] \in M_n$. Then, $\lim_{\varepsilon \to 0} \widetilde{A}_\varepsilon = \widetilde{A}$ and $\lim_{\varepsilon \to 0} A_\varepsilon = A$.

By Theorem 3.1, $\sigma_{\mathcal{K}}(A_\varepsilon) = \{0, \frac{1}{2}\sigma_-(\widetilde{A}_\varepsilon)\}$ and $\sigma_{\mathcal{K}}(-A_\varepsilon) = \{0, -\frac{1}{2}\sigma_+(\widetilde{A}_\varepsilon)\}$, where $\sigma_-(\widetilde{A}_\varepsilon)$ and $\sigma_+(\widetilde{A}_\varepsilon)$ denote the subsets of $\sigma_{\mathbb{R}}(\widetilde{A}_\varepsilon)$ of negative and positive numbers, respectively.

Let $\phi(A_\varepsilon) = \widetilde{B_\varepsilon} \oplus [0]$ and let $\sigma_-(\widetilde{B}_\varepsilon)$ and $\sigma_+(\widetilde{B}_\varepsilon)$ denote the subsets of $\sigma_{\mathbb{R}}(\widetilde{B}_\varepsilon)$ of negative and positive numbers, respectively.

By Theorem 3.1 again, $\sigma_{\mathcal{K}}(\phi(A_\varepsilon)) = \{0, \frac{1}{2}\sigma_-(\widetilde{B}_\varepsilon)\}$ and $\sigma_{\mathcal{K}}(\phi(-A_\varepsilon)) = \{0, -\frac{1}{2}\sigma_+(\widetilde{B}_\varepsilon)\}$. Since $\sigma_{\mathcal{K}}(A_\varepsilon) = \sigma_{\mathcal{K}}(\phi(A_\varepsilon))$, this implies $\sigma_-(\widetilde{A}_\varepsilon) = \sigma_-(\widetilde{B}_\varepsilon)$

and $\sigma_+(\widetilde{A}_\varepsilon) = \sigma_+(\widetilde{B}_\varepsilon)$, that is, $\sigma_{\mathbb{R}}(\widetilde{A}_\varepsilon) \subseteq \sigma_{\mathbb{R}}(\widetilde{B}_\varepsilon)$. Note that it could happen that, although 0 is not an eigenvalue of $\widetilde{A}_\varepsilon$, it is an eigenvalue of $\widetilde{B}_\varepsilon$. Moreover, since the eigenvalues of $\widetilde{A}_\varepsilon$ are all distinct, $\widetilde{B}_\varepsilon$ must have the same (nonzero) real eigenvalues as $\widetilde{A}_\varepsilon$ and these must have multiplicity larger than or equal to 1.

Since $\phi$ is continuous, $\lim_{\varepsilon\to 0}\phi(A_\varepsilon) = \phi(A)$ and, in particular, $\lim_{\varepsilon\to 0}\widetilde{B}_\varepsilon = \widetilde{B}$. Since the eigenvalues of a matrix depend continuously on its entries, we deduce that $\sigma_{\mathbb{R}}(\widetilde{A}) \subseteq \sigma_{\mathbb{R}}(\widetilde{B})$ and that the multiplicity of an eigenvalue $\lambda$ of $\widetilde{A}$ is less than or equal to the multiplicity of $\lambda$ as an eigenvalue of $\widetilde{B}$. Since $\phi$ is invertible and its inverse also preserves the Lorentz-cone spectrum (see Lemma 4.2), by interchanging the roles of $A$ and $\phi(A)$ and considering $\phi^{-1}$ instead of $\phi$, we obtain that $\sigma_{\mathbb{R}}(\widetilde{B}) \subseteq \sigma_{\mathbb{R}}(\widetilde{A})$, counting multiplicities, which completes the proof. □

**4.2.1. The symmetric block-diagonal case.** Utilizing the fact that any matrix $A \in L_n$ is diagonalizable by similarity, we get the following immediate consequence of Lemma 4.5.

**Corollary 4.6.** *Let $\mathcal{M} \in \{D_n, L_n\}$ and let $\phi : \mathcal{M} \to \mathcal{M}$ be a linear map that preserves the Lorentz-cone spectrum. Then $\phi$ preserves the rank, that is, for any $A \in \mathcal{M}$, $A$ and $\phi(A)$ have the same rank. Moreover, the $(n-1)\times(n-1)$ leading principal submatrices of $A$ and $\phi(A)$ also have the same rank.*

From Theorem 2.6 in [3], we obtain the following proposition, which will be used to prove the main result in this section. We denote by $rank(A)$ the rank of a matrix $A \in M_p$. Recall that $S_p$ denotes the subspace of $M_p$ of symmetric matrices.

**Proposition 4.7.** *Let $\psi : S_p \to S_p$ be a bijective linear map preserving rank-one matrices, that is, for every $A \in S_p$, it holds that $rank(\psi(A)) = 1$ whenever $rank(A) = 1$. Then, there is a nonsingular matrix $Q \in M_p$ such that $\psi(A) = QAQ^T$ for any $A \in S_p$.*

We next give the characterization of the Lorentz-cone spectrum preservers defined on $\mathcal{M} \in \{D_n, L_n\}$.

**Theorem 4.8.** *Let $\mathcal{M} \in \{D_n, L_n\}$. A linear map $\phi : \mathcal{M} \to \mathcal{M}$ preserves the Lorentz-cone spectrum if, and only if, there is an orthogonal matrix $Q \in M_{n-1}$ such that*

$$\phi(A) = (Q \oplus [1])A(Q^T \oplus [1]),$$

*for all $A \in \mathcal{M}$. Moreover, if $\mathcal{M} = D_n$, then $Q$ may be taken to be a permutation matrix.*

*Proof.* Sufficiency follows from item (3) of Proposition 2.2. Now, we show necessity.
Assume that $\phi$ preserves the Lorentz-cone spectrum. By Lemma 4.5, for any $A = \widetilde{A} \oplus [a] \in \mathcal{M}$, we have

$$\phi(\widetilde{A} \oplus [a]) = \widetilde{B} \oplus [a] \in \mathcal{M},$$

for some $\widetilde{B} \in M_{n-1}$. Thus, $\phi$ defines a linear map

$$\psi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$$

given by $\psi(\widetilde{A}) = \widetilde{B}$, where $\widetilde{\mathcal{M}} = D_{n-1}$, if $\mathcal{M} = D_n$ and $\widetilde{\mathcal{M}} = S_{n-1}$, if $\mathcal{M} = L_n$. Hence, it is enough to show that there is $Q$ as claimed such that $\psi(\widetilde{A}) = Q\widetilde{A}Q^T$ for any $\widetilde{A} \in \widetilde{\mathcal{M}}$. From Theorem 4.4, $\phi$ is a bijective unital map, implying that so is $\psi$.

**Case 1**: Suppose that $\mathcal{M} = D_n$. Denote by $\widetilde{E}_{ii}$ the $(n-1) \times (n-1)$ leading principal submatrix of $E_{ii}$, $i = 1, \ldots, n-1$. By Lemma 4.5, for $i < n$, $\psi(\widetilde{E}_{ii}) = \widetilde{E}_{jj}$, for some $j < n$. Hence, since $\psi$ is bijective, there is a permutation matrix $Q \in M_{n-1}$ such that $\psi(\widetilde{E}_{ii}) = Q\widetilde{E}_{ii}Q^T$, for all $i = 1, \ldots, n-1$. Since $\widetilde{E}_{ii}$, $i = 1, \ldots, n-1$, is a basis of $D_{n-1}$, we have $\psi(\widetilde{A}) = Q\widetilde{A}Q^T$ for any $\widetilde{A} \in D_{n-1}$.

**Case 2**: Suppose that $\mathcal{M} = L_n$. Since, by Theorem 4.4, $\phi$ is bijective, it follows that so is $\psi$. By Corollary 4.6, $\psi$ preserves the rank of any matrix $\widetilde{A} \in S_{n-1}$. Hence, by Proposition 4.7, there exists an invertible matrix $P \in M_{n-1}$ and $\alpha \in \mathbb{R}\backslash\{0\}$ such that

$$\widetilde{B} = \psi(\widetilde{A}) = \alpha P\widetilde{A}P^T$$

for any $\widetilde{A} \in M_{n-1}$. Since $\phi(I) = I$, we have $\alpha PP^T = I$, which implies $\alpha > 0$. The conclusion follows by taking $Q = \sqrt{\alpha}P$. □

**4.2.2. The general block-diagonal case.** The next result is proved in [4].

**Proposition 4.9.** *Let $T : M_p \to M_p$ be a bijective linear map with the property $T(S) \subseteq S$, where $S$ denotes the set of singular matrices in $M_p$. Then, $T$ is a standard linear map.*

We next give the characterization of the Lorentz-cone spectrum preservers defined on $W_n$.

**Theorem 4.10.** *A linear map $\phi : W_n \to W_n$ preserves the Lorentz-cone spectrum if, and only if, there exists an invertible matrix $P \in M_{n-1}$ such that*

$$\phi(A) = (P \oplus [1])A(P^{-1} \oplus [1])$$

*for all $A \in W_n$, or*

$$\phi(A) = (P \oplus [1])A^T(P^{-1} \oplus [1])$$

*for all $A \in W_n$.*

*Proof.* **Necessity**: Assume that $\phi$ preserves the Lorentz-cone spectrum. Then, by Lemma 4.5, for any $A = \widetilde{A} \oplus [a]$, we have

$$\phi(\widetilde{A} \oplus [a]) = \widetilde{B} \oplus [a]$$

for some $\widetilde{B} \in M_{n-1}$. Thus, $\phi$ defines a linear map

$$\psi : M_{n-1} \to M_{n-1}$$

given by $\psi(\widetilde{A}) = \widetilde{B}$ which, by Lemma 4.5, preserves singularity. Since $\phi$ is bijective, so is $\psi$ and, by Proposition 4.9, there exist invertible matrices $P, Q \in M_{n-1}$ such that

$$\widetilde{B} = \psi(\widetilde{A}) = P\widetilde{A}Q$$

for all $\widetilde{A} \in M_{n-1}$, or

$$\widetilde{B} = \psi(\widetilde{A}) = P\widetilde{A}^T Q$$

for all $\widetilde{A} \in M_{n-1}$. Since $\phi$ is unital, we have that $\psi$ is unital and so, $PQ = I$, completing the proof.

**Sufficiency:** Assume that $\phi$ has one of the claimed forms and let $A = \widetilde{A} \oplus [a]$. By hypothesis we have $\phi(A) = \widetilde{B} \oplus [a]$, with $\widetilde{B}$ being similar to $A$. Recall that any square matrix is similar to its transpose. Now, the conclusion follows from Theorem 3.1. □

### 4.3. Standard Lorentz-cone spectrum preservers on $S_n$

In this section, we restrict our attention to standard linear maps $\phi : S_n \to S_n$ and characterize those that preserve the Lorentz-cone spectrum. We conjecture that, in fact, the family of standard maps includes all the Lorentz-cone spectrum linear preservers on $S_n$.

**Lemma 4.11.** *Let $A = uv^T$, with $u, v \in \mathbb{R}^n$, be a nonzero matrix. Then $A$ is symmetric if, and only if, $v = \alpha u$ for some nonzero scalar $\alpha$.*

*Proof.* The sufficiency part is trivial. Let $A$ be symmetric. Denote by $x_i$ the $i$th coordinate of a given vector $x$. Since $A$ is nonzero, it follows that $u_i v_j \neq 0$ for at some indices $i$, $j$. In particular, $u_i \neq 0$. Then,

$$v_i u = uv^T e_i = Ae_i = A^T e_i = vu^T e_i = u_i v.$$

So, $v = \frac{v_i}{u_i} u$, completing the proof. □

We next give the characterization of the standard maps defined on $S_n$ that preserve the Lorentz-cone spectrum.

**Theorem 4.12.** *Let $\phi : S_n \to S_n$ be a standard linear map. Then, $\phi$ preserves the Lorentz–cone spectrum if and only if there exists and orthogonal matrix $Q \in M_{n-1}$ such that*

$$\phi(A) = (Q \oplus [1])A(Q^T \oplus [1]),$$

*for all $A \in M_n$.*

*Proof.* The "if" claim follows from Proposition 2.2. Now we show the "only if" claim. By hypothesis, there exist matrices $P, Q \in M_n$ such that (4.1) holds. Since $\phi$ preserves the Lorentz-cone spectrum, by Theorem 4.4, $\phi$ is unital and so $Q = P^{-1}$.

For $i \in \{1, \ldots, n\}$ we have

$$\phi(E_{ii}) = PE_{ii}P^{-1},$$

implying that $\phi(E_{ii}) = p_i q_i^T$, where $p_i$ and $q_i^T$ denote, respectively, the $i$th column of $P$ and the $i$th row of $P^{-1}$. Since, by assumption, $\phi(E_{ii})$ is symmetric, by Lemma 4.11, we have $q_i = \alpha_i p_i$ for some $\alpha_i \in \mathbb{R}$. Thus,

$$\begin{aligned}\phi(E_{ii}) &= \alpha_i p_i p_i^T = (\alpha_i \|p_i\|^2) \frac{p_i}{\|p_i\|} \frac{p_i^T}{\|p_i\|} \\ &= \beta_i \begin{bmatrix} \hat{r}_i \hat{r}_i^T & a_i \hat{r}_i \\ a_i \hat{r}_i^T & a_i^2 \end{bmatrix},\end{aligned}$$

where $r_i^T = \frac{1}{\|p_i\|} p_i^T := [\hat{r}_i^T \ a_i]$, with $a_i \in \mathbb{R}$, and $\beta_i = \alpha_i \|p_i\|^2$. Since $1 \in \sigma(E_{ii}) = \sigma(\phi(E_{ii}))$, we have $\beta_i = 1$. Thus, $\alpha_i = \frac{1}{\|p_i\|^2} > 0$.

For $i < n$, we have $\sigma_{\mathcal{K}}(-\phi(E_{ii})) = \sigma_{\mathcal{K}}(-E_{ii}) = \{0, -1/2\}$, where the last equality follows from Theorem 3.1. By Theorem 3.7 and taking into account that $\beta_i = 1$, it follows that $a_i = 0$ (and $\|\hat{r}_i\| = 1$). Thus, the last row of $P$ is a scalar multiple of $e_n^T$, the last row of $I_n$.

For $i = n$, we have $\sigma_{\mathcal{K}}(\phi(E_{nn})) = \sigma_{\mathcal{K}}(E_{nn}) = \{1, 1/2\}$, where the last equality follows from Theorem 3.1 or Corollary 3.5. By Theorem 3.7, we deduce that $\hat{r}_n = 0$ and $\beta_n a_n^2 = 1$. Since $\beta_n = 1$, then $|a_n| = 1$. By a possible multiplication of $P$ and $P^{-1}$ by $-1$, we assume $a_n = 1$. Thus, the last column of $P$ equals $e_n$, the last column of $I_n$.

Hence, we have shown that $P = \widetilde{P} \oplus [1]$, where $\widetilde{P} \in M_{n-1}$ and

$$\widetilde{q}_i = \alpha_i \widetilde{p}_i, \qquad \text{with } \alpha_i = \frac{1}{\|p_i\|^2} = \frac{1}{\|\widetilde{p}_i\|^2}, \tag{4.5}$$

where $\widetilde{p}_i$ and $\widetilde{q}_i^T$ denote the $i$th column and the $i$th row of $\widetilde{P}$ and $\widetilde{P}^{-1}$, respectively.

For $i \in \{1, 2, \ldots, n-1\}$, we have

$$\phi(E_{ni} + E_{in}) = \begin{bmatrix} 0 & \widetilde{p}_i \\ \widetilde{q}_i^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & \widetilde{p}_i \\ \alpha_i \widetilde{p}_i^T & 0 \end{bmatrix}.$$

Since $\phi(E_{ni} + E_{in})$ is symmetric and $\widetilde{p}_i$ is nonzero, we have $\alpha_i = 1$, implying $\widetilde{P}^{-1} = \widetilde{P}^T$. □

### 4.4. Standard Lorentz-cone spectrum preservers on $M_n$

As mentioned in the introduction, the standard linear maps $\phi : M_n \to M_n$ that preserve the Lorentz-cone spectrum were very recently characterized in [8]. When we became aware of that paper, we already had a proof of this result using the techniques developed in this manuscript. So, for completeness and hoping that our approach give some light in proving our conjecture that the family of standard linear maps includes all the Lorentz-cone spectrum linear preservers on $M_n$, we include our proof in this manuscript.

**Lemma 4.13.** *Let $\phi : M_n \to M_n$ be a standard linear map preserving the Lorentzcone spectrum. Then, there is no invertible matrix $P \in M_n$ such that $\phi(A) = PA^T P^{-1}$ for all $A \in M_n$.*

*Proof.* Suppose that there is an invertible matrix $P \in M_n$ such that $\phi(A) = PA^TP^{-1}$ for all $A \in M_n$. Let $i \in \{1, 2, \dots, n-1\}$. Then,

$$\phi(E_{ni}) = PE_{in}P^{-1} = p_i q_n^T = \begin{bmatrix} \hat{p}_i\hat{q}_n^T & b_n\hat{p}_i \\ a_i\hat{q}_n{}^T & a_ib_n \end{bmatrix},$$

where $p_i = [\hat{p}_i^T \ a_i]^T$ and $q_n^T = [\hat{q}_n^T \ b_n]$ denote, respectively, the $i$th column of $P$ and the $n$th row of $P^{-1}$. We have $\sigma_{\mathcal{K}}(\phi(E_{ni})) = \sigma_{\mathcal{K}}(E_{ni}) = [0, 1/2]$, where the second equality follows from Corollary 3.5. Since $E_{in}$ is nilpotent, we have that $\phi(E_{ni})$ is a nilpotent rank-one matrix with infinitely many Lorentz eigenvalues. By Lemma 3.8, $\hat{q}_n \neq 0$ and $\hat{p}_i\hat{q}_n^T = 0$, which implies $\hat{p}_i = 0$. Since $i \in \{1, 2, \dots, n-1\}$ is arbitrary and $P$ is invertible, we get a contradiction. $\square$

We next give the characterization of the standard maps defined on $M_n$ that preserve the Lorentz-cone spectrum.

**Theorem 4.14.** *Let $\phi : M_n \to M_n$ be a standard linear map. Then, $\phi$ preserves the Lorentz–cone spectrum if and only if there exists an orthogonal matrix $Q \in M_{n-1}$ such that*

$$\phi(A) = (Q \oplus [1])A(Q^T \oplus [1]),$$

*for all $A \in M_n$.*

*Proof.* The "if" claim follows from Proposition 2.2. Now we show the "only if" claim. By hypothesis, there exist matrices $P, Q \in M_n$ such that (4.1) or (4.2) hold. Since $\phi$ preserves the Lorentz-cone spectrum, by Theorem 4.4, $\phi$ is unital and so $P$ and $Q$ are invertible and $Q = P^{-1}$. Then, by Lemma 4.13, (4.1) holds.

For $i = 1, 2, \dots, n$, let $p_i = [\hat{p}_i^T \ a_i]^T$ and $q_i^T = [\hat{q}_i^T \ b_i]$ denote, respectively, the $i$th column of $P$ and the $i$th row of $P^{-1}$. For $i < n$, we have

$$\phi(E_{ni}) = p_n q_i^T = \begin{bmatrix} \hat{p}_n\hat{q}_i^T & b_i\hat{p}_n \\ a_n\hat{q}_i{}^T & a_nb_i \end{bmatrix}.$$

Moreover, $\phi(E_{ni})$ is a rank-one nilpotent matrix, as it is similar to $E_{ni}$. Taking into account Corollary 3.5, $\sigma_{\mathcal{K}}(\phi(E_{ni})) = \sigma_{\mathcal{K}}(E_{ni}) = [0, 1/2]$. Thus, by Lemma 3.8, $b_i = 0$, $\hat{p}_n = 0$ and $a_n \neq 0$, implying that the last column of $P$ and, thus, the last column of $P^{-1}$, is a multiple of $e_n$, the last column of $I_n$. Moreover, $b_n = 1/a_n$.

Then, we have

$$\phi(E_{nn}) = p_n q_n^T = \begin{bmatrix} 0 & 0 \\ a_n\hat{q}_n^T & 1 \end{bmatrix}.$$

If $\hat{q}_n \neq 0$, by Theorem 3.4, $\sigma_{\mathcal{K}}(\phi(E_{nn}))$ is infinite, which contradicts the fact, given by Theorem 3.1, that $\sigma_{\mathcal{K}}(\phi(E_{nn})) = \sigma_{\mathcal{K}}(E_{nn}) = \{1, 1/2\}$. Thus, $\hat{q}_n = 0$ and $P = \widetilde{P} \oplus [a_n]$, for some invertible $\widetilde{P} \in M_{n-1}$.

Let $u \in \mathbb{R}^{n-1}$. For

$$A = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \in M_n,$$

we have

$$\phi(A) = \begin{bmatrix} 0 & a_n\widetilde{P}u \\ 0 & 0 \end{bmatrix}.$$

Since $\sigma_{\mathcal{K}}(\phi(A)) = \sigma_{\mathcal{K}}(A)$, by Theorem 3.4, $\|a_n\widetilde{P}u\| = \|u\|$. As $u$ is arbitrary, it follows that $a_n\widetilde{P}$ is an orthogonal matrix.

Finally, we show that $|a_n| = 1$. For $i, j \in \{1, 2, \ldots, n-1\}$, with $i \neq j$, we have

$$\phi(E_{nj} + E_{in}) = \begin{bmatrix} 0 & \frac{1}{a_n}\hat{p}_i \\ a_n\hat{p}_j^T & 0 \end{bmatrix}.$$

Also, $\sigma_{\mathcal{K}}(\phi(E_{nj} + E_{in})) = \sigma_{\mathcal{K}}(E_{nj} + E_{in}) = \{-1/2\}$, where the last equality follows from Corollary 3.5. By Theorem 3.4, taking into account that $\widetilde{p}_j^T\widetilde{p}_i = 0$, it follows that $\|\frac{1}{a_n}\widetilde{p}_i\| = 1$ (note that part 3. (ii) in the theorem applies). Since $\|a_n\widetilde{p}_i\| = 1$, we get $|a_n| = 1$. Then the result in the statement holds with $Q = \widetilde{P}$ if $a_n = 1$ and $Q = -\widetilde{P}$ otherwise. □

## 5. Conclusions

In this paper we study linear maps $\phi : \mathcal{M} \to \mathcal{M}$ that preserve the Lorentz-cone spectrum of any matrix $A \in \mathcal{M}$, for some subspaces $\mathcal{M}$ of $M_n = M_n(\mathbb{R})$, $n \geq 3$. We characterize such maps $\phi$ when (i) $\mathcal{M}$ is the subspace of $M_n$ formed by the block-diagonal matrices that are a direct-sum of an $(n-1) \times (n-1)$ block and a $1 \times 1$ block, which we denote by $W_n$; (ii) $\mathcal{M}$ is the subspace of $W_n$ formed by the diagonal matrices; and (iii) $\mathcal{M}$ is the subspace of $W_n$ of symmetric matrices. In all three cases, we show that these functions are standard linear maps (see Definition 4.10) of some particular form. We then focus on the case $\mathcal{M} = S_n$, the subspace of $M_n$ consisting of the symmetric matrices, and characterize the linear maps $\phi : \mathcal{S}_n \to \mathcal{S}_n$ that are standard and preserve the Lorentz-cone spectrum. An analogous analysis is presented when $\mathcal{M} = M_n$ using our techniques, though this study was performed recently in [8], in a more general context. In contrast with the block-diagonal case, the characterization is the same for both subspaces $S_n$ and $M_n$ and is analogous to the one for the subspace of $W_n$ of symmetric matrices. We conjecture that if a linear map $\phi : \mathcal{M} \to \mathcal{M}$, with $\mathcal{M} \in \{S_n, M_n\}$, preserves the Lorentz-cone spectrum, then it must necessarily be a standard linear map. We hope the techniques developed in this paper can give, in the future, some light on a proof of this conjecture.

As a subproduct of our research, we describe the Lorentz-cone spectrum of some important classes of matrices in $M_n$.

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