Linear preservers of the Lorentz spectrum.
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Prerequisites.

Before the theoretical description of this project is presented, let me mention some important information about what you need to know in order to get involved in this research:

- The math background needed to work on this project is upper-division Linear Algebra.
- Being capable of writing good proofs is a must.
- Having good problem solving skills is desirable. Students involve in this project should be independent and creative.

When you read the description of the project below, you may feel overwhelmed by lots of technical words that you have not heard about before. Do not worry! We will spend some time learning the background in depth before we start working on the project.

Description of the project.

Given a matrix A in M_n, the algebra of $n \times n$ matrices with real entries, and a closed convex cone $K \subseteq \mathbb{R}^n$, the *eigenvalue complementarity problem* consists of finding a scalar $\lambda \in \mathbb{R}$ and a nonzero vector $x \in \mathbb{R}^n$ such that

$$x \in K, \quad Ax - \lambda x \in K^*, \quad x^T(A - \lambda I_n)x = 0,$$

where

$$K^* := \{y \in \mathbb{R}^n : x^T y \geq 0, \ \forall x \in K\}$$

denotes the (positive) dual cone of K. If $K = \mathbb{R}^n$, then the eigenvalue complementarity problem reduces to the usual eigenvalue problem for the matrix A.

The eigenvalue complementarity problem originally arose in the solution of a contact problem in mechanics and has since been used in other applications in physics, economics, and engineering, including, for example, the stability of dynamical systems.

In two recent papers, the complementarity eigenvalue problem associated with the Lorentz cone was considered. The Lorentz cone, for $n \geq 2$, is defined by

$$\mathcal{K}^n := \{(x, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : ||x|| \leq x_n\}.$$

It is also known as the ice-cream cone. By $||x||$ we denote the 2-norm of x. The Lorentz cone is widely used in optimization theory as an instance of a second-order cone, which has special importance in linear and quadratic programming.

It is well known that the Lorentz cone is self-dual, that is, $(\mathcal{K}^n)^* = \mathcal{K}^n$. Therefore, for $A \in M_n$, the eigenvalue complementarity problem relative to $\mathcal{K}^n$ consists of finding a scalar $\lambda \in \mathbb{R}$ and a nonzero vector $x \in \mathbb{R}^n$ such that

$$x \in \mathcal{K}^n, \quad (A - \lambda I)x \in \mathcal{K}^n, \quad x^T(A - \lambda I)x = 0, \tag{1}$$

where I denotes the identity matrix of the appropriate order. It is well-known that (1) always admits a solution.

If a scalar λ and a nonzero vector x satisfy (1), we call λ a *Lorentz eigenvalue* of A and x an associated *Lorentz eigenvector* of A. We call the set of all Lorentz eigenvalues of A the *Lorentz spectrum of* A and denote it by $\sigma_{\mathcal{K}}(A)$. For brevity, we write L-eigenvalue, L-eigenvector, and L-spectrum instead of Lorentz eigenvalue, Lorentz eigenvector, and Lorentz spectrum, respectively.

The roots of the characteristic polynomial of a matrix $A \in M_n$ will be called the *standard eigenvalues* of A, to distinguish them from the L-eigenvalues.

In a recent paper, the authors focused on the problem of studying the linear maps $\phi : W_n \to W_n$ that preserve the L-spectrum, that is, such that $\sigma_{\mathcal{K}}(\phi(A)) = \sigma_{\mathcal{K}}(A)$, for all $A \in W_n$, where W_n is a subspace of M_n, $n \geq 3$. The paper started by characterizing such maps ϕ for the following subspaces W_n of M_n: the subspace of diagonal matrices; the subspace of block-diagonal matrices $\widetilde{A} \oplus [a]$, where $\widetilde{A} \in M_{n-1}$ is symmetric; and the subspace of block-diagonal matrices $\widetilde{A} \oplus [a]$, where $\widetilde{A} \in M_{n-1}$ is a generic matrix. In each of these cases, it was shown that the maps should be what were called *standard maps*, that is, maps of the form $\phi(A) = PAQ$ for all $A \in W_n$ or $\phi(A) = PA^TQ$ for all $A \in W_n$, for some matrices $P, Q \in M_n$. In addition, when W_n is either M_n or the subspace S_n of symmetric matrices in M_n, the standard linear maps $\phi : W_n \to W_n$ that preserve the L-spectrum were described, and it was conjectured that linear maps that are not standard do not preserve the L-spectrum.

In an even more recent paper, written with some of the 2021 REU students, the case $n = 2$ was studied. The goal of this paper was to study the non-trivial problem of characterizing the linear maps $\phi : W_2 \to W_2$ that preserve the L-spectrum, when W_2 is either M_2 or the subspace S_2 of M_2 of symmetric matrices. It was proven that such maps are standard and that, in the case $W_2 = M_2$, their form is less restrictive than the one for $n \geq 3$. The main differentiating feature between the cases $n = 2$ and $n \geq 3$ is that the Lorentz cone in $\mathbb{R}^2$ is *polyhedral*, i.e., it can be expressed as the intersection of a finite number of half-spaces. This implies that the L-spectrum of a matrix in M_2 is always finite, contrary to what happens for matrices of order $n \geq 3$, which can have infinite L-spectrum.

The goal of the summer 2022 project is to extend some results in the paper for $n \geq 2$ to $n \geq 3$.