8.1 Basis and Dimension

Definition A basis for a subspace $V$ of $\mathbb{R}^n$ is a linearly independent spanning set.\(^1\)

Definition The dimension of a subspace $V$ of $\mathbb{R}^n$ is the number of elements in any basis of $V$.

Problem 8.1.1. 1. Find two different bases for the following set and show that they form a basis:

$V := \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

What is the dimension of $V$? Is $V = \mathbb{R}^3$?

If possible, write the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ in each basis.

\(^1\)Please note that one topic not covered in section this week is the change of basis matrix. It was discussed in lecture and covered in the last homework question. Please take a look at it after section, and come see me Friday in office hours if it is giving you trouble.
2. Find two different bases for the following set and show that they form a basis:

\[ V := \text{Span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \end{bmatrix} \right\} \]

What is the dimension of \( V \)? Is \( V = \mathbb{R}^3 \)?

If possible, write the vectors \( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) and \( \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) in each basis.
Problem 8.1.2. Do the following sets form a basis for \( P_2 \)? If so, prove that it is a basis, and if not explain why. If possible write the polynomial \( 1 + t + t^2 \) in terms of each basis.

1. \( S = \{t, 2t - 1, t + 3\} \)

2. \( S = \{t, 2t - 1, t^2 + t + 1\} \)

3. \( S = \{t - 1, 1 - t\} \)

4. \( S = \{t - 1, 1 - t^2, 0, 1 + t + t^2\} \)

5. \( S = \{t^2 + 2t + 1, 1 - t, 2t^2 + t + 1, t^2\} \)
8.2 Kernel and Nullity

Problem 8.2.1. Give the definition of the kernel of a linear transformation.

Problem 8.2.2. Give the definition of the null space of a matrix. What is the nullity of a matrix?

Problem 8.2.3. How are the kernel and null space related?

Definition The column space of $A$ is the subspace spanned by its columns. The rank of a matrix $A$ is the dimension of the column space of $A$. We denote the rank by $\text{rk}(A)$.

Problem 8.2.4. Give the definition of the image of a linear transformation.

Problem 8.2.5. How is the image of a matrix related to the column space of that matrix?

Problem 8.2.6. Why do we have so much language that basically means the same thing?

Definition Suppose that $A$ is an $m \times k$ matrix. The subspace of $\mathbb{R}^k$ spanned by the rows of $A$ is called the row space of $A$, which we denote $\text{row}(A)$.

Problem 8.2.7. Consider the following matrix:

$$A = \begin{bmatrix}
-1 & 2 & 0 & 0 & 0 \\
0 & -1 & -1 & 1 & 1 \\
-1 & 1 & 0 & -1 & 1 \\
1 & -2 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & -1
\end{bmatrix}$$

1. Using the least bit of “math” possible, describe the row space, column space, and kernel of the matrix. (You do not need to do row reduction.) What can you tell about the nullity and rank?
2. Find a basis for the row space, column space, and kernel of the matrix. What can you tell about the nullity and rank?

Problem 8.2.8. True or false? Be sure to explain your answer describing how row reduction affects the given spaces if it does at all and why it doesn’t if that is the case.

1. True or false: Row reduction changes the kernel of a matrix. Explain.

2. True or false: Row reduction changes the columns space of a matrix. Explain.

3. True or false: Row reduction changes the rank of a matrix. Explain.

4. True or false: Row reduction changes the row space of a matrix. Explain.
Problem 8.2.9. For each of the following matrices, find a basis for the kernel and image of the linear transformations they determine.

\[
\begin{bmatrix}
3 & 1 & 0 & 1 \\
2 & 1 & 1 & 1 \\
0 & 1 & -1 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
2 & 3 \\
2 & 3 \\
4 & 6 \\
6 & 9
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]

Problem 8.2.10. Look at the reduced row-echelon forms of each matrix in Problem 8.2.9 above.

1. Describe how you use row reduced echelon form to get the kernel/null space, image/column space, and the row space.

2. Discuss rank and nullity of each in terms of reduced row-echelon form.