The zip car industry is booming in San Francisco! In the beginning there were three locations where you could pick up and drop off zip cars:

- The Airport (SFO)
- Berkeley
- Downtown San Francisco

After one week, only ten percent of the cars from the airport are returned in Berkeley, fifty percent are returned downtown, and the rest are tourists who take them back to the airport when they are finished. Of the cars borrowed from Berkeley, 40% are returned to the same location, and the rest are split evenly between the other locations. A fourth of the cars borrowed from downtown are returned at the airport and twice as many cars are returned to downtown as to Berkeley.

Let $a_0$ represent the number of cars starting at the airport, $b_0$ be the number of cars starting at Berkeley, and $d_0$ be the number of cars starting in downtown San Francisco. Let $a_1$, $b_1$, and $d_1$ be the number of cars in each place after one week.

---

1I will give this to you before the first exam. Put your section day and time and the TA’s name for now or the first two digits of your TARDIS, which is 31 for my 8AM and 32 for my noon section.

2While this is a real problem, I made some of the details up myself.
Prework

Problem 1.1. Draw a diagram that helps you visualize the information about where the zip cars will be in one week.

The following are examples of student solutions:

```
SFO
30% 25% 10% 30%
BK
25% 30% 50% 10%
DT
50% 25% 10% 30%
```

The airport \((a_0)\) 40% 10% 50%
Berkely \((b_0)\) 30% 40% 30%
Downtown \((d_0)\) 25% 25% 50%

\[
\begin{array}{c|c|c|c}
\text{The airport (}a_1\text{)} & \text{Berkeley (}b_1\text{)} & \text{Downtown (}d_1\text{)} \\
\hline
\text{The airport (}a_0\text{)} & 40\% & 10\% & 50\% \\
\text{Berkely (}b_0\text{)} & 30\% & 40\% & 30\% \\
\text{Downtown (}d_0\text{)} & 25\% & 25\% & 50\% \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\text{ }a_1 & \text{ }b_1 & \text{ }d_1 \\
0.4a_0 & 0.1a_0 & 0.5a_0 \\
0.3b_0 & 0.4b_0 & 0.3b_0 \\
0.25d_0 & 0.25d_0 & 0.5d_0 \\
\end{array}
\]
**Problem 1.2.** If there are 100 cars at each location, where will they be after one week?

<table>
<thead>
<tr>
<th></th>
<th>Airport</th>
<th>Berkeley</th>
<th>Downtown</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 = $a_0$</td>
<td>40</td>
<td>10</td>
<td>50</td>
</tr>
<tr>
<td>100 = $b_0$</td>
<td>30</td>
<td>40</td>
<td>30</td>
</tr>
<tr>
<td>100 = $d_0$</td>
<td>25</td>
<td>25</td>
<td>50</td>
</tr>
<tr>
<td>sum</td>
<td>95</td>
<td>75</td>
<td>130</td>
</tr>
</tbody>
</table>

95 cars @ the Airport  
75 cars @ Berkeley  
95 cars @ Downtown

Bonus: write equations that describe where the zip cars will be after one week:

\[
40a_0 + 30b_0 + 25d_0 = a_1 \\
10a_0 + 40b_0 + 25d_0 = b_1 \\
50a_0 + 30b_0 + 50d_0 = d_1
\]

\[
a := 0.4(100) + 0.25(100) + 0.3(100) = 40 + 25 + 30 = 95 \\
b := 0.1(100) + 0.25(100) + 0.4(100) = 10 + 25 + 40 = 75 \\
d := 0.5(100) + 0.5(100) + 0.3(100) = 50 + 50 + 30 = 130
\]

$a_0, b_0, c_0 = 100$, total = 300  
@ airport: 40+30+25=95  
@ Berkeley: 10+40+35=75  
@ Downtown: 50+30+50=130 SUM= 300

**In Class Work**

**Problem 1.1.** Write equations that describe where the zip cars will be after one week.

\[
A_1 = 0.4A_0 + 0.3B_0 + 0.25D_0 \\
B_1 = 0.1A_0 + 0.4B_0 + 0.25D_0 \\
D_1 = 0.5A_0 + 0.3B_0 + 0.5D_0
\]
Problem 1.2. As chief executive, decide how many of your 1,000 cars to place in each location. (Is there an optimal initial placement?) Use SENTENCES to explain what you are doing.

Below is an initial placements chosen by students.
Your answer will depend on the initial placement you chose.
Notice how both initial placements seem to “get closer” after one week.

Initial car placement : \( a_0 = 300 \quad b_0 = 400 \quad d_0 = 300 \) As a vector: \[
\begin{bmatrix}
a_0 \\
b_0 \\
d_0
\end{bmatrix} = \begin{bmatrix}
300 \\
400 \\
300
\end{bmatrix}.
\]

• Where will the cars be after one week? When you have the answer, write it as a vector.

\[
\begin{align*}
A_1 &= 0.4(300) + 0.3(400) + 0.25(300) \\
B_1 &= 0.1(300) + 0.4(400) + 0.25(300) \\
D_1 &= 0.5(300) + 0.3(400) + 0.5(300)
\end{align*}
\]

\[
\begin{bmatrix}
A_1 \\
B_1 \\
D_1
\end{bmatrix} = \begin{bmatrix}
315 \\
265 \\
420
\end{bmatrix}
\]

• Where will the cars be after two weeks? (Let’s call this \( a_2, b_2, \) and \( d_2 \). You get the idea!)

\[
\begin{align*}
A_2 &= 0.4(315) + 0.3(265) + 0.25(420) \\
B_2 &= 0.1(315) + 0.4(265) + 0.25(420) \\
D_2 &= 0.5(315) + 0.3(265) + 0.5(420)
\end{align*}
\]

\[
\begin{bmatrix}
A_2 \\
B_2 \\
D_2
\end{bmatrix} = \begin{bmatrix}
310.5 \\
242.5 \\
447
\end{bmatrix}
\]

Initial car placement : \( a_0 = 325 \quad b_0 = 375 \quad d_0 = 300 \) As a vector: \[
\begin{bmatrix}
a_0 \\
b_0 \\
d_0
\end{bmatrix} = \begin{bmatrix}
325 \\
375 \\
300
\end{bmatrix}.
\]

\[
\begin{align*}
a_1 &= 0.4x325 + 0.3x375 + 0.25x300 \approx 318 \\
b_1 &= 0.1x325 + 0.4x375 + 0.25x300 \approx 257 \\
d_1 &= 0.5x325 + 0.3x375 + 0.5x300 \approx 425
\end{align*}
\]

• Where will the cars be after two weeks? (Let’s call this \( a_2, b_2, \) and \( d_2 \). You get the ideal!)

\[
\begin{align*}
a_2 &\approx 311 \\
b_2 &\approx 241 \\
d_2 &\approx 448
\end{align*}
\]
**Problem 1.3.** How would you know where the cars will be after one year? (Use SENTENCES!) This would be a lot easier to calculate if we could use Wolfram Alpha... If only it could handle such a complicated system of linear equations!

Start with the first week, calculate by using the equations in previous problem and then take the new number and make it the initial to calculate the cars returned next week. Repeat until after 52 weeks.

Plug it in 52 times (for 52 weeks) and use Wolfram Alpha

**Problem 1.4.** Decide with your group how you would know where should you put the cars initially? Is there an optimal starting set up? (Use SENTENCES!)

I’ve meshed the solutions of a couple of groups. The second group actually found a solution to the zip car problem!

You start with the equation that will tell you how many cars are placed after one week. Then the optimal way is when your initial equals your ending $a_0 = a_1 = a$, $b_0 = b_1 = b$, $c_0 = c_1 = c$.

\[
\begin{align*}
a_0 &= 0.4a_0 + 0.3b_0 + 0.25d_0 \\
b_0 &= 0.1a_0 + 0.4b_0 + 0.25d_0 \\
d_0 &= 0.5a_0 + 0.3b_0 + 0.5d_0
\end{align*}
\]

because (subtract $a$, $b$, and $d$ from both sides)

\[
\begin{align*}
0 &= 0.4a + 0.3b + 0.25d - a \\
0 &= 0.1a + 0.4b + 0.25d - b \\
0 &= 0.5a + 0.3b + 0.5d - d
\end{align*}
\]

This group found the equations for an optimal solution. Then solved the system by scaling and adding equations to eliminate variables. This is equivalent to the row reduction technique you learned later this week.

\[
\begin{align*}
a &= 0.4a + 0.3b + 0.25d \\
b &= 0.1a + 0.4b + 0.25d \\
d &= 0.5a + 0.3b + 0.5d
\end{align*}
\]

\[
\begin{align*}
-60a + 30b + 25d &= 0 \\
50a + 30b - 50d &= 0 \\
10a - 60b + 25d &= 0
\end{align*}
\]

\[
\begin{align*}
-70a + 90b &= 0 \\
110a - 75d &= 0
\end{align*}
\]

\[
\begin{align*}
b &= \frac{2}{9}a \\
d &= \frac{110}{75}
\end{align*}
\]