**Bonus Prework on Vector Spaces**

May 28, 2013

**Instructions**: Print out this page, and write directly on it, using the back if necessary. Do not staple. This is due under my office door Tuesday May 21st by 5PM. Come see me in my office hours if you have questions. You do not need to answer questions in the footnotes.

**Problem 1.** Which of the following are vector spaces? Why or why not? If it is a vector space, give a basis.

(a) Everywhere you can get on just the magic carpet from Project 2.

**Solution 1.** Everywhere you can get on the magic carpet (span \(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\)) is a vector space. It is a line in \(\mathbb{R}^2\).

Some students spelled it out that “Everywhere you can get on the magic carpet is

\[
W = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} = \left\{ x \begin{bmatrix} 1 \\ 2 \end{bmatrix} | x \in \mathbb{R} \right\}
\]

and spelled out

\[
x \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ y \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in W \text{ and } x, y, c \in \mathbb{R}.
\]

You would need to add this to get full 6 of 6 marks on this type of question. Remember the interesting thing about subspaces and vector spaces is that if you add two things in the set you get a new thing in the set. This comment goes for all of them.

(a) The span \(\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}\) is a subset of \(\mathbb{R}^2\).

(b) Because this lines goes through origin, the zero vector is included.

(c) To check addition and multiplication: Since we are in the span \(\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}\), we have \(x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (x + y) \begin{bmatrix} 1 \\ 2 \end{bmatrix}\), which belongs to the span \(\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}\), because is a linear combination of the vector \(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\).

(d) If we were to multiply the vector \(x \begin{bmatrix} 1 \\ 2 \end{bmatrix}\) by any scalar, \(c \left( x \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)\) is also included in span \(\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}\), because is a multiple of \(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\). Therefore, it is a vector space, and its basis is \(\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}\).

(b) Everywhere you can get on the magic carpet, hoover board, and flying unicorn from Project 3.

**Solution 2.** Span is the set of all linear combinations of \(\begin{bmatrix} 1 \\ 1 \\ 6 \\ 1 \\ 3 \\ 8 \\ 1 \\ 4 \end{bmatrix}\), \(\begin{bmatrix} 6 \\ 3 \\ 8 \\ 1 \\ 1 \\ 6 \end{bmatrix}\), \(\begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}\)

\[
W = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 6 \\ 1 \\ 3 \\ 8 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 8 \\ 1 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} \right\} = \left\{ x \begin{bmatrix} 1 \\ 1 \\ 6 \\ 1 \\ 3 \\ 8 \\ 1 \\ 4 \end{bmatrix} + y \begin{bmatrix} 6 \\ 3 \\ 8 \\ 1 \\ 1 \\ 6 \end{bmatrix} + z \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} | x, y, z \in \mathbb{R} \right\}
\]
This is a subset of \( \mathbb{R}^3 \), the zero vector exists because if \( x = y = z = 0 \), there is a zero vector. The space is closed under addition and scalar multiplication.

\[
\vec{w} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} + z \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} \in W
\]

\[
c\vec{w} = cx \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + cy \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} + cz \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} \in W
\]

\[
\vec{v} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} + c \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}
\]

\[
\vec{w} + \vec{v} = (a + x) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (b + y) \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} + (c + z) \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} \in W
\]

(c) \( \{p(t) = at^2 + bt + c | p(0) = 1\} \)

**Solution 3.** In this case the zero vector is \( p(t) = 0 \). This cannot be a vector space because it does not have the zero vector. For \( p(t) = 0 \) that means \( a = b = c = 0 \), and \( 0t^2 + 0t + 0 \). This condition does not satisfy the zero vector, because we get \( p(0) = 1 \), where \( a(0) + b(0) + c = 1 \). \( c \) needs to be equal to 1. \( 0t^2 + 0t + 0 \neq a(0)^2 + b(0) + 1 \).

**Solution 4.** This is not a vector space because it does not contain the zero vector. The origin for polynomials means \( 0t^2 + 0t + 0 = 0 \). Since \( p(0) = 1 \), it means we can never get the zero vector.

(d) A matrix, \( A \) is antisymmetric if \( A^T = -A \). Is the set \( S \) of antisymmetric matrices a vector space?

\[
S = \{ A \in M_{2\times2} | A^T = -A \}
\]

**Solution 5.** We assume \( A \) equals \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). If \( A^T = -A \), we get

\[
\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix},
\]

so

\[
\begin{align*}
a &= -a = 0 \\
b &= -c \\
c &= b \\
d &= d = 0
\end{align*}
\]

So we assume \( A \) to be \( x \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \). This space satisfies addition because for any matrices

\[
w \left( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) + v \left( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) = (w + v) \left( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right),
\]

which is still antisymmetric. The space also satisfies scalar multiplication, because if we multiply by a scalar, we get

\[
c \left( x \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) = (cx) \left( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right),
\]

which is just a multiple of the original 2x2 antisymmetrical matrix, and is included in \( S \). For \( x = 0 \) we get that zero vector \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) is also included in \( S \). Therefore, \( S \) is a vector space and the basis is \( \{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \} \).

(e) The set of colors what can be displayed by a computer using the RPG color model.

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1How would this problem be different if we change the condition to \( p(1) = 0 \)?
2This question was on an exam in 2011. A similar question that was also on an exam that year, if you want more practice, is “What about the set of symmetric matrices where \( A^T = A \)?” What about the set of upper triangular 2 \( \times \) 2 matrices? Do the matrices with determinant 0 form a vector space? What about determinant 1?
Solution 6. The set of colors is not a vector space because it is not closed under scalar multiplication. If you multiply the RGB vector by something out of scope, you will not get a color. For example, if $c=2$

\[
c\vec{r} = 2 \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 200 \\ 0 \\ 0 \end{bmatrix}.
\]

Since the intensity cannot be bigger than 200, it fails.

(f) The solution set (find it) to the following system of linear equations:

\[
\begin{align*}
x_1 + & 2x_2 + 3x_3 + 4x_4 = 0 \\
2x_1 + & 4x_2 + 6x_3 + 7x_4 = 0
\end{align*}
\]

Solution 7.

\[
\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\
2 & 4 & 6 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\
0 & 0 & 0 & 1 & 0 \end{bmatrix}.
\]

\[W = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.
\]

closed under addition:

\[
x \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + A \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + B \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} = (x + A) \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (y + B) \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in W
\]

closed under multiplication:

\[
c \left( x \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) = (cx) \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (cy) \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in W
\]

zero vector, $x = 0, y = 0$:

\[
0 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0} \in W
\]

basis:

\[
\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.
\]

\[\text{How would this be different if the right hand side of each equation were equal to 1?}\]